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WHY IT IS IMPOSSIBLE TO TRISECT AN ANGLE OR TO CONSTRUCT A REGULAR POLYGON OF 7 OR 9 SIDES BY RULER AND COMPASSES

PROFESSOR LEONARD EUGENE DICKSON

UNIVERSITY OF CHICAGO

1. *Introduction.*—An alert student of geometry who has learned how to bisect any angle is apt to ask if every angle can be trisected, and, if not, why not. After learning how to construct regular polygons of 3, 4, 5, 6, 8 and 10 sides, he will be inquisitive about the missing ones of 7 and 9 sides. The teacher will be in a comfortable position if he knows the facts and what is involved in the simplest discussion to date of these questions.

There is a valid reason why such a discussion is omitted from texts on geometry, which record simple constructions which have been found to be successful. It should be expected to be beyond a beginner to follow a proof that a proposed problem for construction is impossible, since such a proof must cover the infinite number of conceivable constructions, one of which might be successful.

Accordingly our first step is to formulate our geometric problems in algebraic language. For each of the problems, as well as for the famous problem of the duplication of a cube, we shall be led to a cubic equation. Our final step is to prove and apply a general theorem concerning cubic equations.

2. *Duplication of a cube.*—We take as the unit of length a side of the given cube, and seek the length x of a side of the required cube whose volume shall be double the volume of the given cube. Hence,

$$x^3=2. \tag{1}$$

3. *Trisection of an angle.*—Given an angle A and hence the

$(A/3)$. We employ the well-known trigonometric identity

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}.$$

Multiply each member by 2 and write x for $2\cos(A/3)$. Thus

$$x^3 - 3x = 2\cos A.$$

We are to prove that an *arbitrary** angle A can not be trisected by ruler and compasses. It suffices to prove this for the angle $A = 120°$. Then $\cos A = -\tfrac{1}{2}$ and the cubic equation becomes

$$x^3 - 3x = -1. \tag{2}$$

4. *Regular polygons of 7 and 9 sides.*—The angle at the center subtended by a side of a regular polygon of 9 sides is $360°/9 = 40°$. The construction of an angle of 40° is equivalent to the trisection of angle 120°, which, as we just saw, depends upon the cubic (2).

The construction of a regular polygon of 7 sides is equivalent to the construction of an angle B containing 360/7 degrees and hence to the construction of a line of length $\cos B$. Since $7B = 360°$, $\cos 3B = \cos 4B$. But

$$\cos 3B = 4\cos^3 B - 3\cos B,$$
$$\cos 4B = 2\cos^2 2B - 1 = 2(2\cos^2 B - 1)^2 - 1.$$

Writing x for $2\cos B$, we get

$$2\cos 3B = x^3 - 3x,\quad 2\cos 4B = (x^2 - 2)^2 - 2.$$

Equating these values, we obtain

$$o = x^4 - x^3 - 4x^2 + 3x + 2 = (x - 2)(x^3 + x^2 - 2x - 1).$$

But $x = 2$ would give $\cos B = 1$, whereas B is acute. Hence

$$x^3 + x^2 - 2x - 1 = 0. \tag{3}$$

5. *Our cubic equations have no rational roots.*—Suppose that equation (2), for example, has the rational root n/d, length $\cos A$ of the adjacent leg of a right triangle whose

* Certain angles, like 180° and 90°, can be trisected. For $A = 180°$, $A/3 = 60°$, $\cos A = -1$, $\cos(A/3) = \tfrac{1}{2}$, and our equation becomes $x^3 - 3x = -2$, and has the root $x = 1$. The fact that 180° can be trisected is explained by the fact that the cubic equation has the *rational* root 1.

hypotenuse is the unit of length, we seek a line of length cos where n and d are whole numbers without a common factor >1. Then

$$\frac{n^3}{d^3} - 3\frac{n}{d} = -1, \qquad \frac{n^3}{d} = 3nd - d^2,$$

so that n^3/d is a whole number. Thus, if $d \neq \pm 1$, n and d have a common divisor >1, contrary to hypothesis. Hence $d = \pm 1$ and the root is a whole number. But if a root x of (2) is a whole number, it divides the left member of (2), giving the quotient $x^2 - 3$, which is a whole number; hence it must divide the right member, -1, so that $x = 1$ or -1. By trial, neither 1 nor -1 is a root of (2). Hence (2) has no rational root.

This discussion applies also to equation (3). In the case of equation (1), we must try also the divisors 2 and -2 of the constant term. We thus see that *no one of the cubic equations* (1), (2), (3) *has a rational root.*

6. Theorem.—*It is not possible to construct by ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root.*

We begin by investigating the nature of a positive number p such that a line of length p can be constructed by ruler and compasses. The ends of this line, as well as other points located in the course of the construction, are determined as the intersections of auxiliary straight lines and circles. Consider the equations of these lines and circles referred to a fixed pair of rectangular axes, the y-axis not being parallel to any of our straight lines. The equation of any one of our lines is

$$y = mx + b. \tag{4}$$

Let $y = m'x + b'$ be another of our lines which intersects (4). The coördinates of their point of intersection are

$$x = \frac{b' - b}{m - m'}, \qquad y = \frac{mb' - m'b}{m - m'},$$

which are rational functions of the coefficients of the two lines.

To find the coördinates of the intersections of (4) with the circle

$$(x - e)^2 + (y - f)^2 = r^2$$

of radius r and center (e, f), we eliminate y and obtain a quadratic equation for x. Thus x, and hence also y, involves no irrationality other than a square root in addition to the irrationalities which already occurred in m, b, e, f, r.

Finally, the intersections of two circles are given by the intersections of one of them with their common chord, so that this case reduces to the preceding.

Hence the coördinates of the various points located by the construction, and therefore also the length p of the segment joining two of them, are found by a finite number of rational operations and extractions of real square roots, performed upon rational numbers or upon numbers obtained by such operations.

For example, a side of a regular pentagon inscribed in a circle of radius unity is of length

$$\frac{1}{2}\sqrt{10-2\sqrt{5}}. \tag{5}$$

This point settled, consider a cubic equation with rational coefficients and having a root x_1 such that a line of length x_1 or $-x_1$ (according as x_1 is positive or negative) can be constructed by ruler and compasses. We shall prove that one of the roots of the cubic is rational. Suppose that x_1 is irrational, since otherwise there is nothing to be proved.

Since a line of length $\pm x_1$ can be constructed, we saw that x_1 is obtained by rational operations and extractions of real square roots. It may involve superimposed radicals like (5). A two-story radical, like (5), which is not expressible rationally in terms of a finite number of square roots of rational numbers, is called a radical of *order* 2. But

$$\sqrt{4-2\sqrt{3}} = \sqrt{3}-1$$

is a radical of order 1, not 2, and similar simplifications are assumed to be made whenever possible. In general, an n-story radical, having n superimposed radicals, is said to be of *order* n if it is not expressible rationally in terms of radicals each with fewer than n superimposed radicals.

Finally, if x_1 involves $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{15}$, we replace $\sqrt{15}$ by $\sqrt{3}\cdot\sqrt{5}$. Similarly, if $x_1 = r - 7s$, where r has the value (5) and

$$s = \frac{1}{2}\sqrt{10+2\sqrt{5}},$$

so that $rs = \sqrt{5}$, we agree to express x_1 in the simpler form $x_1 = r - 7\sqrt{5}/r$, involving a single radical of order 2.

In view of these agreements, no one of the radicals of highest order n in x_1 equals a rational function with rational coefficients of the remaining radicals of order n and radicals of lower order, while no one of the radicals of order $n-1$ equals a rational function of the remaining radicals of order $n-1$ and radicals of lower order, etc.

Let $\sqrt{k}$ be a radical of highest order n in x_1. Then

$$x_1 = \frac{a + b\sqrt{k}}{c + d\sqrt{k}},$$

where a, b, c, d do not involve $\sqrt{k}$, but may involve other radicals of order n, as well as radicals of lower order. If $d = 0$,

(6) $$x_1 = e + f\sqrt{k} \qquad (f \neq 0),$$

where neither e nor f involves $\sqrt{k}$. If $d \neq 0$, we multiply the numerator and denominator of our fractional expression for x_1 by $c - d\sqrt{k}$, which is not zero since $\sqrt{k} \neq c/d$ in view of our agreements above. Hence whether $d = 0$ or $d \neq 0$, we obtain (6).

By hypothesis, x_1 is a root of a cubic equation

(7) $$x^3 + \alpha x^2 + \beta x + \gamma = 0$$

with rational coefficients α, β, γ. Expanding the powers in

(8) $$(e + f\sqrt{k})^3 + \alpha(e + f\sqrt{k})^2 + \beta(e + f\sqrt{k}) + \gamma,$$

replacing the square of $\sqrt{k}$ by k, and collecting terms, we evidently obtain an expression of the form $A + B\sqrt{k}$, where neither A nor B involves $\sqrt{k}$. Hence $A + B\sqrt{k} = 0$. If $B \neq 0$, we would have $\sqrt{k}$ expressed as a rational function $-A/B$ of the radicals other than $\sqrt{k}$ in x_1, contrary to our agreement. Hence $B = 0$, therefore $A = 0$. Since (8) reduced to $A + B\sqrt{k}$,

$$(e - f\sqrt{k})^3 + \alpha(e - f\sqrt{k})^2 + \beta(e - f\sqrt{k}) + \gamma$$

reduces to $A - B\sqrt{k} = 0$. In other words,

(9) $$x_2 = e - f\sqrt{k}$$

is also a root of our cubic equation (7). It is distinct from the root (6) since $f \neq 0$. As is well known, the sum of the three roots of (7) equals $-a$. Hence the third root is

$$(10) \qquad x_3 = -a - x_1 - x_2 = -a - 2e.$$

By hypothesis, a is rational. We shall prove that e is also rational, so that the root x_3 is rational, and our theorem will be proved.

Suppose that e is irrational. Since e is a component part of a number (6) which can be constructed by ruler and compasses, we know that it involves no irrationalities other than real square roots. Let, therefore, $\sqrt{s}$ be one of the radicals of highest order in e. Applying to e the argument which led to (6), we have $e = e' + f' \sqrt{s}$ and hence, by (10),

$$x_3 = g + h\sqrt{s} \qquad (h \neq 0),$$

where neither g nor h involves $\sqrt{s}$. By the argument which led to (9), we see that $g - h\sqrt{s}$ is a root, distinct from x_3, of our cubic (7). Since a cubic can not have four distinct roots, this fourth root must be identical with x_1 or x_2. Thus

$$(11) \qquad e \pm f\sqrt{k} = g - h\sqrt{s}.$$

But $\sqrt{s}$ and all the radicals appearing in g, h and s occur in $x_3 = -a - 2e$, and hence occur in e, since a is rational. Hence (11) enables us to express $\sqrt{k}$ rationally in terms of the radicals occurring in e and f. This is contrary to the hypothesis made concerning x_1 and $\sqrt{k}$.

7. *Conclusion.*—We saw that each of the problems to duplicate a cube, trisect an angle, and construct a regular polygon of 7 or 9 sides, each by ruler and compasses, led to a cubic equation (1), (2) or (3) with rational coefficients and having no rational root. It follows from our preceding theorem that each of these problems is impossible.

It can be proved by a suitable modification of the above discussion that it is impossible to construct by ruler and compass a regular polygon of n sides if n is divisible by any prime not of the form $2^k + 1$ or if n is divisible by the square of any prime exceeding 2. The only possible values of n are therefore $2^l\, p_1\, p_2 \ldots$, where $p_1, p_2, \ldots$ are distinct primes 3, 5,

17, 257, . . . of the form 2^k+1. Conversely, when n is of this necessary form, the polygon can be constructed. A proof of this theorem is given in the writer's article in *Monographs on Modern Mathematics*, Longmans, Green and Co., 1911. Since it is desirable that such important results should be proved by at least two wholly different methods, it is in place to cite the very brief, but more advanced, proof by group theory in Miller, Blichfeldt, and Dickson's *Theory and Applications of Finite Groups*, Wiley and Sons, 1916, pp. 320–6.

COLLEGE ENTRANCE REQUIREMENTS IN MATHEMATICS.

A Preliminary Report: The National Committee on Mathematical Requirements.

General Considerations.—The committee aims to formulate standard minimum requirements adapted to the needs of academic and engineering colleges, and other institutions of similar grade. Such further requirements as may be appropriate to particular colleges or classes of colleges have been discussed in an earlier report, dealing with elective high school mathematics.

The primary purpose of college entrance requirements is to test the candidate's ability to benefit by college instruction. This ability depends—as far as our present inquiry is concerned—on (1) general intelligence, intellectual maturity and mental power; (2) specific knowledge and training required as preparation for the various courses of the college curriculum.

Mathematical ability appears to be a sufficient, but not a necessary, condition for general intelligence.* For this, as well as for other reasons, it would appear that *college entrance requirements in mathematics should be formulated primarily on the basis of the special knowledge and training required for the successful study of courses which the student will take in college.*

The separation of prospective college students from the others in the early years of the secondary school is neither feasible nor desirable. It would therefore seem to be obvious that secondary school courses for the first two years can not be planned with specific reference to college entrance requirements. Fortunately there appears to be no real conflict of interest between those students who ultimately go to college and those who do not, as far as mathematics is concerned. It

* A recent investigation made by the Department of Psychology at Dartmouth College showed that all students of high rank in mathematics had a high rating on general intelligence; the converse was not true, however.

will be made clear in what follows that a course in this subject, covering from two to two and one-half years, in a standard four-year high school, so planned as to give the most valuable mathematical training which the student is capable of receiving, will provide adequate preparation for college work.

The Selection of Topics.—In this selection preparation for college courses in mathematics need not be specifically considered. Not all college students study mathematics; it is therefore reasonable to expect college departments in this subject to adjust themselves to the previous preparation of their students. Nearly all college students do, however, study one or more of the physical sciences (astronomy, physics, chemistry) and one or more of the social sciences (history, economics, political science, sociology). Entrance requirements must therefore insure adequate mathematical preparation for these subjects. Moreover, it may be assumed that adequate preparation for these two groups of subjects will be sufficient for all other subjects the preparation for which may be expected from the secondary schools.

A recent investigation made by the National Committee gives valuable information on the question under discussion. A number of college teachers, prominent in their respective fields, were asked to assign to each of the topics in the following table its value as preparation for the elementary courses in their respective subjects. Table I gives a summary of the replies, arranged in two groups—"Physical Sciences," including astronomy, physics and chemistry, and "Social Sciences," including history, economics, sociology and political science.

TABLE I.

VALUE OF TOPICS AS PREPARATION FOR ELEMENTARY COLLEGE COURSES.

In the headings of the following table, E = essential, C = of considerable value, S = of some value, O = of little or no value, N = number of replies received. The figures in the first four columns of each group are percents of the number of replies received.

	Physical Sciences					Social Sciences				
	E	C	S	O	N	E	C	S	O	N
Negative numbers—their meaning and use	79	5	10	5	39	45	17	22	17	18
Imaginary numbers—their meaning and use	23	21	25	31	39	13	13	37	37	16
Simple formulas—their meaning and use	93	5	2		41	47	26	21	5	19

	Physical Sciences					Social Sciences				
	E	C	S	O	N	E	C	S	O	N
Graphic representation of statistical data	57	25	15	3	40	57	24	14	5	21
Graphs (mathematical and empirical):										
(*a*) as a method of representing dependence	62	16	22		37	15	54	15	15	13
(*b*) as a method of solving problems.	46	20	28	6	25	18	18	46	18	11
The linear function, $y = mx + b$	78	14	8		37	29	29	14	29	14
The quadratic function, $y = ax^2 + bx + c$	59	21	17	3	34	8	8	33	50	12
Equations: Problems leading to—										
Linear equations in one unknown	98	2			41	40	7	20	33	15
Quadratic equations in one unknown	78	15	5	2	40	31	8	8	54	13
Simultaneous linear equations in two unknowns	71	24	3	3	38	33	8		58	12
Simultaneous linear equations in more than two unknowns	43	29	23	6	35	8	8	17	67	12
One quadratic and one linear equation in two unknowns	40	24	27	9	33		9	9	82	11
Two quadratic equations in two unknowns	31	19	28	22	32		9		91	11
Equations of higher degree than the 2d	10	32	32	26	31			9	91	11
Literal equations (other than formulas).	43	18	32	7	28		10	40	50	10
Ratio and proportion	84	8	3	5	39	37	26	32	5	19
Variation	50	13	20	17	30	17	33	25	25	12
Numerical computation:										
With approximate data—rational use of significant figures	61	36		3	39	40	27	20	13	12
Short-cut methods	27	38	24	10	37	29	35	23	12	17
Use of logarithms	62	29	7	2	42	12	29	29	29	17
Use of other tables to facilitate computation	24	45	26	5	38	18	29	41	12	17
Use of slide rule	24	39	26	12	38	11	39	28	22	18
Theory of exponents	36	31	25	8	36		21	21	57	14
Theory of logarithms	34	26	21	18	38	7	13	20	60	15
Arithmetic progression	16	32	38	13	37	23	29	12	35	17
Geometric progression	19	27	40	14	37	23	25	18	35	17
Binomial theorem	35	32	19	13	37	13	20	27	40	15
Probability	9	32	41	19	32	20	35	35	10	20
Statistics:										
Meaning and use of elementary concepts	23	28	31	17	29	55	36	5	5	22
Frequency distributions and frequency curves	15	19	35	32	26	47	33	10	10	21
Correlation	11	18	39	32	28	33	47	14	5	21
Numerical Trigonometry:										
Use of sine, cosine, and tangent in the solution of simple problems involving right triangles	68	21	3	8	38			25	75	12

	Physical Sciences					Social Sciences				
	E	C	S	O	N	E	C	S	O	N
Demonstrative geometry	68	15	12	6	34		21	43	36	14
Plane trigonometry (usual course)	57	27	11	5	37	8	23	31	38	13
Analytic geometry:										
Fundamental conceptions and methods in the plane	32	45	19	3	31		15	38	46	13
Systematic treatment of—										
Straight line	34	37	20	9	35	9	9	18	64	11
Circle	29	43	20	9	35		18	9	73	11
Conic sections	18	41	26	15	34		9	18	73	11
Polar coordinates	18	26	41	15	34			18	82	11
Empirical curves and fitting curves to observations	12	38	38	12	34	8		25	67	12

TABLE II.

Topics in Order of Value as Preparation for Elementary College Courses.

The figures in the column headed "E" are taken from Table I, taking in each case the higher of the two "E" ratings there given. The column headed "E + C" gives in each case the sum of the two ratings for "E" and "C." An asterisk indicates that the topic in question is now included in the definitions of the College Entrance Examination Board.*

	E	E + C
*Linear equations in one unknown	98	100
Simple formulas—their meaning and use	93	98
*Ratio and proportion	84	92
*Negative numbers—their meaning and use	79	84
*Quadratic equations in one unknown	78	93
The linear function: $y = mx + b$	78	92
*Simultaneous linear equations in two unknowns	71	95
Numerical trigonometry—the use of the sine, cosine, and tangent in the solution of simple problems involving right triangles	68	89
*Demonstrative geometry	68	83
Use of logarithms in computation	62	91
Computation with approximate data—rational use of significant figures	61	97
*Graphs as a method of representing dependence	61	78
The quadratic function: $y = ax^2 + bx + c$	59	80
Plane trigonometry—usual course	57	84

* The list includes all of the requirements of the College Entrance Examination Board except those relating to algebraic technique. The topic of "Negative numbers" has also been given an asterisk as it is clearly implied though not explicitly mentioned in the C. E. E. B. definitions.

	E	E + C
Graphic representation of statistical data	57	82
Statistics—meaning and use of elementary concepts	55	91
Variation	50	63
Statistics—frequency distributions and curves	47	80
*Graphic solution of problems	46	66
*Simultaneous linear equations in more than two unknowns	43	72
*Literal equations	43	61
*Simultaneous equations, one quadratic, one linear	40	64
*Theory of exponents	36	67
*Binomial theorem	35	67
Analytic geometry of the straight line	34	71
Theory of logarithms	34	60
Statistics—correlation	33	80
*Simultaneous quadratic equations	31	50
Analytic geometry of the circle	29	72
Short-cut methods of computation	29	65
Use of tables in computation (other than logarithms)	24	69
Use of slide rule	24	63
*Arithmetic progression	23	52
*Geometric progression	23	48
Imaginary numbers	23	44
Probability	20	55
Conic sections	18	59
Polar coordinates	18	44
Empirical curves and fitting curves to observations	12	50
Equations of higher degree than the second	10	42

The high value attached to the following topics is significant:

Simple formulas—their meaning and use.
The linear and quadratic functions and variation.
Numerical trigonometry.
The use of logarithms and other topics relating to numerical computation.
Statistics.

These all stand well above such standard requirements as—

Arithmetic and geometric progression.
Binomial theorem.
Theory of exponents.
Simultaneous equations involving one or two quadratic equations.
Literal equations.

These results would seem to indicate that a modification of present requirements is desirable from the point of view of college teachers in departments other than mathematics. It is interesting to note how closely the modifications suggested by this inquiry correspond to the modifications in secondary school mathematics foreshadowed by the study of the needs of the high school pupil irrespective of his possible future college attendance. The preliminary reports of the National Committee on "The Reorganization of the First Courses in Secondary School Mathematics" and on "Junior High School Mathematics" recommend that functional relationship be made the "underlying principle of the course," that the meaning and use of simple formulas be emphasized, that more attention than hitherto be given to numerical computation (especially to the methods relating to approximate data), and that work on numerical trigonometry and statistics be included. These recommendations have received widespread approval throughout the country. That they should be in such close accord with the desires of college teachers as to entrance requirements is striking. We find here the justification for the belief expressed earlier in this report that there is no real conflict between the needs of students who ultimately go to college and those who do not.

The Attitude of the Colleges.—Mathematical instruction in this country is at present in a period of transition. While a considerable number of our most progressive schools have for several years given courses embodying most of the recommendations contained in the reorganization report of the National Committee above referred to, the vast majority of schools are still continuing the older type of courses or are only just beginning to introduce modifications. The movement toward reorganization is strong, however, throughout the country, not only in the standard four-year high schools, but also in the newer junior high schools.

During this period of transition it should be the policy of the colleges, while exerting a desirable steadying influence, to help the movement toward a sane reorganization. In particular, they should take care not to place obstacles in the way of changes which are clearly in the interest of more effective

college preparation, as well as of better general education. College entrance requirements will continue to exert a powerful influence on secondary school teaching. Unless they reflect the spirit of sound progressive tendencies, they will constitute a serious obstacle.

In the present report revised definitions of plane geometry and elementary algebra are presented. So far as plane geometry is concerned, the problem of definition is comparatively simple. The proposed definition of the requirement in plane geometry does not differ from the one now in effect under the College Entrance Examination Board. A list of propositions and constructions has, however, been prepared and is included for the guidance of teachers and examiners.

In elementary algebra a certain amount of flexibility is obviously necessary, both on account of the quantitative differences among colleges and of the special conditions attending a period of transition. The former differences are recognized by the proposal of a minor and a major requirement in elementary algebra. The second of these includes the first and is intended to correspond with the two-unit rating of the College Entrance Examination Board. It includes certain optional topics which should afford the latitude needed particularly during the transition period, and should also meet the needs of colleges preferring a requirement intermediate in quantity between one unit and two.

In connection with this matter of units, the committee wishes particularly to disclaim any emphasis on a special number of years or of hours. The unit-terminology is doubtless too well established to be quite ignored in formulating college entrance requirements, but the standard definition of unit* has never been precise, and will now become much less so with the inclusion of the newer six-year program. A time allotment of four or five hours per week in the seventh grade can certainly not have the same weight as the same number of hours in the

* The following definition, formulated by the National Committee on Standards of Colleges and Secondary Schools, has been given the approval of the C. E. E. B. "A unit represents a year's study in any subject in a secondary school, constituting approximately a quarter of a full year's work. A four-year secondary school curriculum should be regarded as representing not more than sixteen units of work."

twelfth grade, and the disparity will vary with different subjects. *What is really important is the amount of subject-matter and the quality of work done in it.* The "unit" can not be anything but a crude approximation to this. The distribution of time in the school program should not be determined by any arbitrary unit scale.

As a further means of securing reasonable flexibility, the committee recommends, in case its proposed definitions should be adopted by the College Entrance Examination Board, that for a limited time—say, five years—the option be offered between examinations based on the old and on the new definitions, so far as differences between them may make this desirable.

In view of the changes taking place at the present time in mathematical courses in secondary schools and the fact that college entrance requirements should as soon as possible reflect desirable changes and assist in their adoption, the National Committee recommends that either the American Mathematical Society or the Mathematical Association of America (or both) maintain a permanent committee on College Entrance Requirements in Mathematics, such a committee to work in close coöperation with other agencies which are now or may in the future be concerned in a responsible way with the relations between colleges and secondary schools.

Definition of College Entrance Requirements.

Elementary Algebra.

Minor Requirement (one unit).

The meaning and use (including the necessary transformations) of simple formulas involving ideas with which the student is familiar and the derivation of such formulas from rules expressed in words.

The dependence of one variable upon another. Numerous illustrations and problems involving the linear function $y = mx + b$. Illustrations and problems involving the quadratic function $y = kx^2$.

The graph and graphic representations in general—their construction and interpretation—including the representation

of statistical data and the use of the graph to exhibit dependence.

Positive and negative numbers—their meaning and use.

Linear equations in one unknown quantity; their use in solving problems.

Simultaneous linear equations involving two unknown quantities; their use in solving problems.

Ratio, as a case of simple fractions; proportion, without the theorems on alternation, etc., and simple cases of variation, without the use of the symbol for variation.

The essentials of algebraic technique. This should include

- The four fundamental operations.
- Factoring of the following types:
  - Monomial factors and simple cases by grouping.
  - The difference of two squares.
  - Trinomials of the second degree (including the square of a binomial) that can be easily factored by trial.
- Fractions, including complex fractions of a simple type.
- Exponents and radicals. The laws for positive integral exponents and the meaning and use of fractional and negative exponents, but not the formal theory. The consideration of radicals may be confined to the simplification of expressions of the form $\sqrt{a^2b}$ and $\sqrt{a/b}$ and to the evaluation of simple expressions involving the radical sign. A process for extracting the square root of a number should be included, but not the process for extracting the square root of a polynomial.

Major Requirement (two units).

In addition to the minor requirement, as specified above, the following:

- Illustrations and problems involving the quadratic function $y = ax^2 + bx + c$.
- Quadratic equations in one unknown; their use in solving problems.
- The use of logarithmic tables in computation, without the formal theory.

Four of the following topics:

Numerical trigonometry—the definitions of the sine, cosine, and tangent of an angle and their use in solving problems involving right triangles. The use of three or four place tables of such functions.

Elementary statistics—including a knowledge of the fundamental concepts and simple frequency distributions, with graphic representations of various kinds.

The binomial theorem for positive integral exponents less than 8; with applications, such as compound interest.

The formula for the nth term and the sum of n terms of arithmetic and geometric progressions, with applications.

Simultaneous linear equations in three unknown quantities and simple cases of simultaneous equations involving one or two quadratic equations—their use in solving problems.

Drill in algebraic manipulation should be limited, particularly in the minor requirement, by the purpose of securing a thorough understanding of important principles and facility in carrying out those processes which are fundamental and of frequent occurrence either in common life or in the subsequent courses that a substantial proportion of the pupils will study. Skill in manipulation must be conceived of throughout as a means to an end, not as an end in itself. Within these limits, skill and accuracy in algebraic technique are of prime importance, and drill in this subject should be extended far enough to enable students to carry out the fundamentally essential processes accurately and with reasonable speed.

The consideration of literal equations, when they serve a significant purpose, such as the transformation of formulas, the derivation of a general solution (as of the quadratic equation) or the proof of a theorem, is important. As a means for drill in algebraic technique they should be used sparingly.

The solution of problems should offer opportunity throughout the course for considerable arithmetical and computational work. The conception of algebra as an extension of arithmetic should be made significant both in numerical applications and in elucidating algebraic principles. Emphasis should be placed on the use of common sense and judgment in computing from approximate data, especially with regard to

the number of significant figures retained, and on the necessity for checking the results. The use of tables to facilitate computation (such as tables of squares and square roots, interest, trigonometric functions, etc.) should be encouraged.

Plane Geometry.

One unit.

The usual theorems and constructions of good text-books, including the general properties of plane rectilinear figures; the circle and the measurement of angles; similar polygons; areas; regular polygons and the measurement of the circle.

The solution of numerous original exercises, including locus problems. Applications to the mensuration of lines and plane surfaces.

The scope of the required work in plane geometry is indicated by the List of Fundamental Propositions and Constructions, which is appended to this report. This list indicates in Section I the type of proposition which, in the opinion of the committee, may be assumed without proof or given informal treatment. Section II contains 52 propositions and 19 constructions which are regarded as so fundamental that they should constitute the common minimum of all standard courses in plane geometry. Section III gives a list of subsidiary theorems which suggests the type of additional propositions that should be included in such courses.

College Entrance Examinations.—College entrance examinations exert in many schools—and especially throughout the eastern section of the country—an influence on secondary school teaching which is very far-reaching. It is, therefore, well within the province of the National Committee to inquire whether the prevailing type of examination in mathematics serves the best interests of mathematical education and of college preparation.

The reason for the almost controlling influence of entrance examinations in the schools referred to is readily recognized. Schools sending students to Harvard, Yale, Princeton, and the larger colleges for women, or to any institution where examinations form the only or prevailing mode of admission, inevitably

direct their instruction toward the entrance examination. This remains true even if only a small percentage of the class intends to take these examinations, the point being that the success of a teacher is often measured by the success of his or her students in these examinations.

In the judgment of the committee, the prevailing type of entrance examination in algebra is primarily a test of the candidate's skill in formal algebraic manipulation. It has not been an adequate test of his understanding or of his ability to apply the principles of the subject. Moreover, it is quite generally felt that the difficulty and complexity of the formal manipulative questions, which have appeared on recent papers set by colleges and such agencies as the College Entrance Examination Board, has often been excessive. As a result, teachers preparing pupils for these examinations have inevitably been led to devote an excessive amount of time to drill in algebraic technique, without insuring an adequate understanding of the principles involved. Far from providing the desired facility, this practice has tended to impair it. For "practical skill, modes of effective technique, can be intelligently, non-mechanically used only when intelligence has played a part in their acquisition." (Dewey, "How We Think," p. 32.)

Moreover, it must be noted that authors and publishers of text-books are under strong pressure to make their content and distribution of emphasis conform to the prevailing type of entrance examination. Teachers in turn are too often unable to rise above the text-book. An improvement in the examinations in this respect will cause a corresponding improvement in text-books and in teaching.

On the other hand, the makers of entrance examinations in algebra can not be held solely responsible for the condition described. Theirs is a most difficult problem. Not only can they reply that as long as algebra is taught as it is examinations must be largely on technique,* but they can also claim with considerable force that technical facility is the only phase of algebra that can be fairly tested by an examination; that a

* The vicious circle is now complete: Algebra is taught mechanically because of the character of the entrance examination; the examination, in order to be fair, must conform to the character of the teaching.

candidate can rarely do himself justice amid unfamiliar surroundings and with a time limit ahead on questions involving real thinking in applying principles to concrete situations; that we must face here a real limitation on the power of an examination to test attainment. Many, perhaps most, teachers will agree with this claim. Past experience is on their side; no effective "power test" in mathematics has as yet been devised, and, if devised, it might not be suitable for use under conditions prevailing during an entrance examination.

But if it is true that the power of an examination is thus inevitably limited, the wisdom and fairness of using it as the sole means of admission to college is surely open to grave doubt. That many unqualified candidates are admitted under this system is not open to question. Is it not probable that many qualified candidates are at the same time excluded? If the entrance examination is a fair test of manipulative skill only, should not the colleges use additional means of learning something about the candidate's other abilities and qualifications?

Some teachers believe that an effective "power test" in mathematics is possible. Efforts to devise such a test should receive every encouragement.

In the meantime, certain desirable modifications of the prevailing type of entrance examinations are possible. The College Entrance Examination Board has recently appointed a committee to consider this question, and a conference* on this subject has been held by representatives of the College Entrance Examination Board, members of the National Committee and others. The following recommendations are taken from the report of the committee just referred to:

"Fully one third of the questions should be based on topics of such fundamental importance that they will have been thoroughly taught, carefully reviewed and deeply impressed by effective drill. They should be of such a degree of difficulty

* At this conference the following vote was unanimously passed: "Voted, That the results of examinations (of the College Entrance Examination Board), be reported by letters A, B, C, D, E and that the definition of the groups represented by these letters should be determined in each year by the distribution of ability in a standard group of papers representing widely both public and private schools.

that any pupil of regular attendance, faithful application and even moderate ability may be expected to answer them satisfactorily.

"There should be both simple and difficult questions testing the candidate's ability to apply the principles of the subject. The early ones of the easy questions should be really easy for the candidate of good average ability who can do a little thinking under the stress of an examination; but even these questions should have genuine scientific content.

"There should be a substantial question which will put the best candidates on their mettle, but which is not beyond the reach of a fair proportion of the really good candidates. This question should test the normal workings of a well-trained mind. It should be capable of being thought out in the limited time of the examination. It should be a test of the candidate's grasp and insight—not a catch question or a question of unfamiliar character making extraordinary demands on the critical powers of the candidate, or one the solution of which depends on an inspiration. Above all, this question should lie near to the heart of the subject, as all well-prepared candidates understand the subject.

"As a rule, a question should consist of a single part and be framed to test one thing—not pieced together out of several unrelated and perhaps unequally important parts.

"Each question should be a substantial test on the topic or topics which it represents. It is, however, in the nature of the case impossible that all questions be of equal value.

"Care should be used that the examination be not too long. The examiner should be content to ask questions on the important topics, so chosen that their answers will be fair to the candidate and instructive to the readers; and beyond this merely to sample the candidate's knowledge on the minor topics."

In addition, the National Committee suggests the following principles:

The examination as a whole should, as far as practicable, reflect the principle that algebraic technique is a means to an end, and not an end in itself.

Questions that require of the candidate skill in algebraic

manipulation beyond the needs of actual application should be used very sparingly.

An effort should be made to devise questions which will fairly test the candidate's understanding of principles and his ability to apply them. These should involve a minimum of manipulative complexity.

In geometry, examinations should be definitely constructed to test the candidate's ability to draw valid conclusions rather than his ability to memorize an argument.

The National Committee has published separately a report on Mathematical Terms and Symbols. It is hoped that examining bodies will avoid the use of terms and symbols not recommended by that report.

Plane Geometry.

List of Fundamental Propositions and Constructions.

I. Assumptions and Theorems for Informal Treatment.

This list contains propositions which may be assumed without proof (postulates) and theorems which it is permissible to treat informally. Some of these propositions will appear as definitions in certain methods of treatment. Moreover, teachers should feel free to require formal proofs of some, if they desire to do so. The precise wording given is not essential, nor is the order in which the propositions are here listed. The list should be taken as representative of the type of propositions which may be assumed, or treated informally, rather than as exhaustive.

1. Through two distinct points it is possible to draw one straight line, and only one.
2. A line segment may be produced to any desired length.
3. The shortest path between two points is the line segment joining them.
4. One and only one perpendicular can be drawn through a given point to a given straight line.
5. The shortest distance from a point to a line is the perpendicular distance from the point to the line.
6. From a given center and with a given radius one and only one circle can be described in a plane.

7. A straight line intersects a circle in at most two points.

8. Any figure may be moved from one place to another without changing its shape or size.

9. All right angles are equal.

10. If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.

11. Equal angles have equal complements and equal supplements.

12. Vertical angles are equal.

13. Two lines perpendicular to the same line are parallel.

14. Through a given point not on a given straight line, one straight line, and only one, can be drawn parallel to the given line.

15. Two lines parallel to the same line are parallel to each other.

16. The area of a rectangle is equal to its base times its altitude.

II. Fundamental Theorems and Constructions.

It is recommended that theorems and constructions, other than originals, to be proved on entrance examinations be chosen from the following list. Originals and other exercises should be capable of solution by direct reference to one or more of these propositions and constructions. It should be obvious that any course in geometry capable of giving adequate training must include considerable additional material. The order here given is not intended to signify anything as to the order of presentation. It should be clearly understood that certain of the statements contain two or more theorems, and that the precise wording is not essential. The committee favors entire freedom in statement and sequence.

A. *Theorems.*

1. Two triangles are congruent if*

   (*a*) two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other;

* Teachers should feel free to separate this theorem into three distinct theorems and to use other phraseology for any such proposition. For example in 1, "Two triangles are equal if . . . ," "A triangle is determined

(*b*) two angles and a side of one are equal, respectively, to two angles and the corresponding side of the other;

(*c*) The three sides of one are equal, respectively, to the three sides of the other.

2. Two right triangles are congruent if the hypotenuse and one other side of one are equal, respectively, to the hypotenuse and another side of the other.

3. If two sides of a triangle are equal, the angles opposite these sides are equal, and conversely.*

4. The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line segment joining them.

5. The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.

6. When a transversal cuts two parallel lines, the alternate-interior angles are equal; and conversely.

7. The sum of the angles of a triangle is two right angles.

8. A parallelogram is divided into congruent triangles by either diagonal.

9. Any (convex) quadrilateral is a parallelogram

(*a*) if the opposite sides are equal;

(*b*) if two sides are equal and parallel.

10. If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.

11. (*a*) The area of a parallelogram is equal to the base times the altitude.

(*b*) The area of a triangle is equal to one half the base times the altitude.

(*c*) The area of a trapezoid is equal to half the sum of its bases times its altitude.

by . . . ,'' etc. Similarly in 2, the statement might read: ''Two right triangles are congruent if, beside the right angles, any two parts (not both angles) in the one are equal to corresponding parts of the other.''

* It should be understood that the converse of a theorem need not be treated in connection with the theorem itself, it being sometimes better to treat it later. Furthermore a converse may occasionally be accepted as true in an elementary course if the necessity for proof is made clear, the proof to be given later.

(*d*) The area of a regular polygon is equal to half the product of its apothem and perimeter.

12. (*a*) If a straight line is drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally.

(*b*) If a line divides two sides of a triangle proportionally, it is parallel to the third side. (Proofs for commensurable cases only.)

(*c*) The segments cut off on two transversals by a series of parallels are proportional.

13. Two triangles are similar if

(*a*) they have two angles of one equal, respectively, to two angles of the other;

(*b*) they have an angle of one equal to an angle of the other and the including sides are proportional;

(*c*) their sides are respectively proportional.

14. If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.

15. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

16. Polygons are similar, if they can be decomposed into triangles which are similar and similarly placed; and conversely.

17. The bisector of an (interior or exterior) angle of a triangle divides the opposite side (produced if necessary) into segments proportional to the adjacent sides.

18. The areas of two similar triangles (or polygons) are to each other as the squares of any two corresponding sides.

19. In any right triangle the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles, each similar to the given triangle.

20. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

21. In the same circle or in equal circles, if two arcs are equal, their central angles are equal; and conversely.

22. In any circle two angles at the center are proportional to their intercepted arcs. (Proof for commensurable case only.)

23. In the same circle or in equal circles, if two arcs are equal their chords are equal; and conversely.

24. (*a*) A diameter perpendicular to a chord bisects the chord and the arcs of the chord.
(*b*) A diameter which bisects a chord (that is not a diameter) is perpendicular to it.

25. The tangent to a circle at a given point is perpendicular to the radius at that point; and conversely.

26. In the same circle or in equal circles, equal chords are equally distant from the center; and conversely.

27. An angle inscribed in a circle is equal to half the central angle having the same arc.

28. Angles inscribed in the same segment are equal.

29. If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon and tangents at the points of division form a regular circumscribed polygon.

30. The circumference of a circle is equal to $2\pi r$. (Informal proof only.)

31.* The area of a circle is equal to πr^2. (Informal proof only.)

The treatment of the mensuration of the circle should be based on related theorems concerning regular polygons, but it should be informal as to the limiting processes involved. The aim should be an understanding of the concepts involved, so far as the capacity of the pupil permits.

B. *Constructions.*

1. Bisect a line segment and draw the perpendicular bisector.
2. Bisect an angle.
3. Draw a perpendicular to a given line through a given point.
4. Construct an angle equal to a given angle.
5. Through a given point draw a straight line parallel to a given straight line.
6. Construct a triangle, given
   (*a*) the three sides;

* The total number of theorems given in this list when separated as will probably be found advantageous in teaching, this number including the converses indicated, is 52.

(*b*) two sides and the included angle;

(*c*) two angles and a side.

7. Divide a line segment into parts proportional to given segments.

8. Given an arc of a circle, find its center.

9. Circumscribe a circle about a triangle.

10. Inscribe a circle in a triangle.

11. Construct a tangent to a circle through a given point.

12. Construct the fourth proportional to three given line segments.

13. Construct the mean proportional between two given line segments.

14. Construct a triangle (polygon) similar to a given triangle (polygon).

15. Construct a triangle equal to a given polygon.

16. Inscribe a square in circle.

17. Inscribe a regular hexagon in a circle.

III. Subsidiary List of Propositions.

The following list of propositions is intended to suggest some of the additional material referred to in the introductory paragraph of Section II. It is not intended, however, to be exhaustive—indeed, the Committee feels that teachers should be allowed considerable freedom in the selection of such additional material, theorems, corollaries, originals, exercises, etc., in the hope that opportunity will thus be afforded for constructive work in the development of courses in geometry.

1. When two lines are cut by a transversal, if the corresponding angles are equal, or if the interior angles on the same side of the transversal are supplementary, the lines are parallel.

2. When a transversal cuts two parallel lines, the corresponding angles are equal, and the interior angles on the same side of the transversal are supplementary.

3. A line perpendicular to one of two parallels is perpendicular to the other also.

4. If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.

5. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.

6. The sum of the angles of a convex polygon of n sides is 2 (n—2) right angles.

7. In any parallelogram
   (*a*) the opposite sides are equal;
   (*b*) the opposite angles are equal;
   (*c*) the diagonals bisect each other.

8. Any (convex) quadrilateral is a parallelogram if
   (*a*) the opposite angles are equal;
   (*b*) the diagonals bisect each other.

9. The medians of a triangle intersect in a point which is two thirds of the distance from the vertex to the mid-point of the opposite side.

10. The altitudes of a triangle meet in a point.

11. The perpendicular bisectors of the sides of a triangle meet in a point.

12. The bisectors of the angles of a triangle meet in a point.

13. The tangents of a circle from an external point are equal.

14.* (*a*) If two sides of a triangle are unequal, the greater side has the greater angle opposite it; and conversely.
   (*b*) If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second; and conversely.
   (*c*) If two chords are unequal, the greater is at the less distance from the center; and conversely.
   (*d*) The greater of two minor arcs has the greater chord; and conversely.

15. An angle inscribed in a semi-circle is a right angle.

16. Parallel lines, tangent to or cutting a circle, intercept equal arcs on the circle.

* Such inequality theorems as these are of importance in developing the notion of dependence or functionality in geometry. The fact that they are placed in our "Subsidiary List of Propositions" should not imply that they are considered of less educational value than those in List II. They are placed here because they are not "fundamental" in the same sense that the theorems of List II are fundamental.

17. An angle formed by a tangent and a chord of a circle is measured by half the intercepted arc.

18. An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.

19. An angle formed by two secants, or by two tangents, to a circle is measured by half the difference between the intercepted arcs.

20. If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.

21. Parallelograms, or triangles, of equal bases and altitudes are equal.

22. The perimeters of two regular polygons of the same number of sides are to each other as their radii, and also as their apothems.

COMMENTS ON THE TEACHING OF GEOMETRY.

FRANK C. TOUTON,

PROFESSOR OF SECONDARY EDUCATION, UNIVERSITY OF CALIFORNIA, BERKELEY.

Purpose or Problem.—The writer of this paper has each day occasion to visit classes where geometry recitations are in progress in the course of his regular supervision work. Some outstanding features of the recitation work are here mentioned, that good classroom practices may become more common, and that practices of doubtful value may be called in question and, if found lacking, abandoned. Certain methods of work which seem effective in the study-recitation process are here considered in some detail and are recommended for trial and for adoption by those who succeed through their use.

Recitation Routine.—Within a given period a record was made of the work in progress at the time of the visit. This report shows that in a large majority of the cases some pupil was reading aloud the work of explanation which he had previously written in full at the board. In many cases the pupil, though standing, had maintained his position at or near his desk and was looking at and "talking at" a figure and written material some fifteen or twenty feet distant.

It is fair to assume that the teachers in charge must have given their approval to the method of explanation in progress. If so, then many teachers believe that good will somehow result from a re-reading of the material placed on the board. On the assumption that explanations are necessary and worth the time commonly devoted to them, it follows that such explanations should be given as effectively as possible.

Now effective presentation demands of the one who gives an explanation that he identify himself with the product offered. This identification of one's self with the product is made possible through taking a position near the diagram and by means of a pointer designating each line, point, angle, or surface referred to in the discussion. Some of the principles involved in good salesmanship should be practiced in the geometry classroom. Each pupil must be made to feel that he is selling his

method and work to his fellows. In the sale of his solution he should use such methods of appeal as will guarantee that each member of the class can see the part of the figure to which attention is directed. In practice, the figures used are often so small, and so located at the board as to be out of the range of vision of many members of the class. The figures used are commonly exact duplicates, letters and all, of those given in the text in use. No salesman would attempt to sell a shop-worn article. It must be new. The element of newness can be used if the figure used in the explanation presents the idea of the exercise in a different form or position. Then, too, a salesman shows but one article at a time and definitely directs the attention of the buyer to the essential qualities of the article he holds.

Again, it is relatively infrequent that one is privileged to observe a pupil take his place at the board and draw as he talks a large figure representing in an effective way each of the elements referred to in the theorem, and then to indicate his general plan of solution, saying, "I propose to prove that these two lines are equal by showing that they are corresponding parts of congruent triangles, or that they each equal a given line," etc. Instead, if the figure is not already at the board, in common practice the pupil silently draws the complete figure while the class awaits his pleasure. Such a practice teaches pupils to waste time. The real purpose of drawing the several lines is not evident to the class, and since no general plan of work has been announced, the completed diagram presents a needlessly complex situation to the pupils who really are in need of help.

How much better it would be were such construction lines as are used in explanations added in the development of the general plan of work which has been announced. The whole explanation would then be full of purpose and would be written more permanently into the nervous system of the pupil, for by original nature we are more responsive to such situations as involve movement than to those which are fixed and static. Since the plane of appeal involves motor neurone centers, the learning thus gained is more permanent.

High-grade Service in Geometry Teaching.—It is the con-

viction of the writer that the highest grade of service rendered by the geometry teacher is that given in problem situations where the pupil is making his first attack upon new theorems and exercises into which situations the teacher enters to guarantee that right habits of thought and work are learned and used. Good habits of thinking in geometry demand: (1) that the facts and relations of the given situation be clearly identified and represented; (2) that the direct outcomes of the given elements and required facts be noted; (3) that a search be made for such combinations of given elements as give promise of usefulness in reaching a situation in which the desired fact or relation will become evident; (4) that care be exercised in calling up and using theorems already proved and other known facts, to the end that the desired conclusion will be reached from known facts which have been properly related; and finally (5) that the materials of the proof be so arranged as to show clearly the correct reason for each step of the proof, with an evident logical arrangement of all materials used throughout the proof.

Where direction (teaching) of the type described above is not given when new theorems and problems are attacked, then the time of the teacher and class is largely consumed in rehearsing explanation given by pupils who have succeeded in their home work. Some teachers have discovered that a greater number of successes are experienced by pupils when at least half of the class period is spent with pupils as they make their first attack on new theorems and problems. Such teaching is vital, interesting to pupils, and stimulating to the teacher, for the needed help in developing and executing plans of attack can be given, and at the proper point.

In a very real sense pupils learn to do by doing under direction. They learn through succeeding in their work. If successes are experienced, learning will necessarily result. Many teachers find it necessary at the beginning of the pupil's work in geometry to guide pupils as they use each of the five steps listed in the previous paragraph. Later it is found necessary to direct the pupils through steps 1–4 only. Later still, through steps 1–3 only; and finally, when correct habits of thinking have been formed, little guidance need be given, ex-

cept in very difficult problem situations. There will always be need for guidance on the part of some pupils, though the need for this direct guidance should be reduced to the minimum.

Where the lesson assignment has been inadequate—that is, where certain of the steps listed above should have been taken under the direction of the teacher and were not taken—then a large portion of the time of the teacher and class must be consumed in the uninteresting routine of explaining solutions and correcting errors. Such explanations must be given and the time of the class must be thus utilized, unless *real teaching* is given, in making adequate lesson assignments and in directing the first work of pupils as they attack their new problem situations. Many teachers of geometry in Wisconsin have caught the vision of the better use of the class period. It has been the delight of the writer to visit a considerable number of classes where *geometry is really taught.*

In good teaching of geometry motivated drills must be given and difficulties must be cleared up for those who have such. Certain problems must from time to time be worked out in full in lesson assignments to guarantee that correct habits of thought and of work are being utilized. An inventory of the advance lesson must be taken.

In preparing to assign the lesson, the teacher will either take sufficient time to work through the several exercises to be assigned or he will have made use of such teaching helps as are provided. It is the conviction of the writer that the study necessary for the teaching of a lesson should be given in preparation for the assignment of that lesson. Only under such conditions can adequate lesson assignments be made.

The following rule may safely be followed: Give as much weight in thought, in time, and in practice to pupils as they attack new situations as you do in rehearsing the details of old situations. Where such a division of the class period is made, due consideration will be given to the teaching of pupils in methods of attack upon new situations and in forming the most profitable habits of thought and of work. In short, pupils will be taught how to think through the guidance given by the teacher in problem situations which require real thinking.

Then, too, sufficient time will be available for checking up the work of pupils and for the giving of help to those who need it.

Lesson Assignments Based on a Recognition of Individual Differences.—Teachers agree in theory at least that each pupil should be given enough to do to tax his total mental capacity to the full for the time allotted to the subject for class work and home study. Knowing as we do that some pupils can do several times as many exercises as others in a given time, the ideal would be reached only through individual instruction where each pupil progresses at his own maximum rate. Our plan for class instruction makes such a provision for lesson assignments well nigh impossible.

Again, we might conceive of our classes as sectioned into given groups on the score of ability so as to place in the failure, inferior, medium, superior and excellent groups, 5 per cent., 20 per cent., 50 per cent., 20 per cent. and 5 per cent, respectively, and then to plan our lesson assignments so as to adapt them to the needs of each of the five groups. Such a plan, too, is doubtless impracticable for the small and median high school, at least for the present time. We can, however, and I think we should, so plan the extent of the lesson assignments as to lay out a certain amount of work, theorems and exercises, for the lower 75 per cent. of the class which we may designate as *required work*, and offer in addition a certain amount of work (number of exercises) for the superior and excellent pupils, which work we, in lesson assignments, might designate as *optional*. The place of the pupil in the upper 25 per cent. of the class clearly depends upon his willingness to do and his success in doing the optional work.

The emphasis given previously in this article on lesson assignments relates, of course, to the work here designated as required. Little direction, if any, should be given to the optional work as far as lesson assignment is concerned. This optional work should be voluntary and largely self-directed. Reports on this optional work when made to the class afford much interest and inspiration to the entire class.

Reviews and Drills.—Good teaching demands that such subject matter as is worth remembering be called up before it has dropped below the level of consciousness. Psychologists show

us that the most rapid drop in the curve of forgetting occurs in the intervals which elapse immediately following the learning of the content. Since the evidence is available and reliable, it follows that good teaching should include provision for effective drills and reviews. Such drills should be quickly carried out and frequently given.

In a few cases the writer has found that the figures for the theorems have been drawn with ink on heavy 24 x 36 inch paper. These figures are drawn carefully with black waterproof ink and differ materially, though in no essential particulars, from the figures given for the theorems in the text in use. The figures are, of course, lettered differently and are filed away in the teacher's cabinet, where they are ready for almost instant use. The use of these figures is recommended for reviews only. Their use will more than double the amount of material or topics which can be reviewed in a given period.

Thinking with Pupils on Original Work.—The writer recommends that from time to time teachers of geometry select original exercises with the solution of which they are not familiar, and that they set for themselves the tasks of writing out the solutions of the problems, making a written record of each idea, hint, or suggestion that is hit upon in the course of the solution. Such a record would in some cases be a striking revelation of the method of trial and success as the only method available in new problem-solving situations. These solutions have been found particularly interesting and helpful when carried out in line with the five steps previously mentioned in this article. In any case, the teacher who takes the trouble to make such a written record of the workings of his own mind in new problem situations will better appreciate the difficulties with which pupils are constantly surrounded.

Teachers find that the written record referred to above serves as a splendid guide in teaching pupils how to succeed in solving original exercises of sufficient difficulty to tax to the full their total mental equipments. The solving of originals by the teacher, both at home and in the classroom, gives the only adequate preparation for the teaching of pupils in the methods of attack and solution of new problem-solving situations.

SOME IDEALS IN TEACHING MATHEMATICS.

EDWIN W. SCHREIBER,

HEAD OF MATHEMATICS DEPARTMENT, PROVISO TOWNSHIP HIGH SCHOOL, MAYWOOD, ILL.

I. In my early teaching I shall try to attain the following objectives of preparation and practice:

1. To have some historical background as to the growth of the modern high school, so that I may better interpret my function as a teacher.

2. To familiarize myself with the immediate and ultimate aims of education, so that I may be convincing in my arguments when an issue is raised by a colleague or patron.

3. To avoid as much waste as possible by using the principles of business management.

4. To reduce to routine as many factors in my teaching as will allow such procedure.

5. To preserve order to the best of my ability.

6. To realize that the content of a course must vary with the needs of different communities and different social groups, and yet not in my inexperience to presume to know too much about what the various groups need.

7. To be careful not to over-emphasize my own subject (mathematics) and cause my students to think they are drinking from the only true source of the spring of knowledge.

8. To know something of the different types of learning involved in high school subjects, so that I may better understand the learning process.

9. To show motor skill along some line that I shall use in my teaching (in the case of mathematics the ability to make while before my class a good-looking drawing, both free-hand and with instruments). I can not expect the pupils to do a piece of work that I can not do successfully myself. The power of example is very convincing.

10. To study the laws of associating symbols.

11. To assist my pupils in reflective thinking, for this truly is the major portion of my job.

12. To cultivate for myself habits of harmless enjoyment, and thus be in a position to assist my students to the same end.

13. To require each pupil under my care to make an honest effort to express himself correctly and forcibly. We have a tendency in the classroom to allow too often a poorly chosen word or a slovenly expression.

14. To place my pupils in the proper frame of mind, so that the most benefit may result from the recitation.

15. To learn early in my career that coöperation is the keynote of success, and that I should therefore spend a great deal of time and energy and thought in trying to see the other fellow's point of view and in assisting the general progress of the institution which claims me as one of its workmen.

II. In my later teaching, when I have had the opportunity of reviewing my path with some sense of perspective, and thus appreciating the evident gaps, I shall try to make that path in the future more continuous, more symmetrical, and more regular by applying the following principles:

1. To make of myself more or less an authority upon some particular branch of my work, so that I may gain the inspiration that comes only through research. To continually follow the beaten path is deadening.

2. To be sure to remember that perhaps nine-tenths of what I teach will soon be forgotten by my students and to make the one-tenth they do remember *count.*

3. To affiliate myself with the professional organizations which foster progress in my subject. Great benefit may be derived from such relationships because of the wonderful opportunity for the exchange of ideas.

4. To subscribe to the leading journals in my subject, and thus be in touch with the latest achievements of the thinkers in my profession.

5. To build up my own reference library, not with the idea of completeness, for the public libraries should be used freely, but with the thought that with careful selection I may possess as my very own some of the best works that have been written. Great satisfaction comes with the mere sense of owning a good book.

6. To be willing to assist where I may in furthering the cause of my subject—be it as a member of some committee, as an officer of an organization, or as the speaker on the program of an organization. In other words, to reach out beyond my own school and help in the big general issues.

7. To study scientifically the measurements of the results of teaching. To familiarize myself with statistical procedure, so that I can better interpret the results of my teaching and present the same more forcibly and economically to my superior officer.

8. To have a working knowledge of the standardized tests in my field and to be able to discuss them critically with my colleagues.

9. To attend at frequent intervals the summer sessions of good colleges or universities, and thus advance myself in the knowledge of my subject or the methods of presenting the same; *both* are essential to a successful teacher.

10. To take good care of my body and soul, that I may have faith in my job and happiness in it.

COMPUTATION IN JUNIOR HIGH SCHOOL MATHEMATICS.

D. W. WERREMEYER,

HEAD OF DEPARTMENT OF MATHEMATICS, WEST TECHNICAL HIGH SCHOOL, CLEVELAND, OHIO.

With the present tendency of including so much geometric and algebraic material in the junior high school and of devoting so much time to informational subjects in many vocations and avocations, one may well ask the question, "Is there sufficient time left to give the necessary drill and practice to make pupils proficient in the four fundamental operations with integral numbers and with common and decimal fractions?"

The writer does not wish to find any fault with or to throw any discouragement upon the useful work in geometry and algebra that is being advocated in these grades, for it is, indeed, the hope of solving the junior high school problem in mathematics, but he rather desires to sound a warning against carrying any movement to an extreme.

With the lack of adequate *constructive supervision* in junior high school mathematics, there is great danger of teachers' becoming so highly enthused about the multiplicity of informational subjects that are now being advocated for the curriculum in mathematics that pupils may be left woefully weak in arithmetic skill, the very point that needs most careful attention, because it is upon this point that schools meet adverse criticisms on every hand in the business and professional world. If the new courses of study do not make ample provision for this phase of the work, it is evident that the pendulum will soon swing back and that much of our hope of progress will be lost.

The writer contends that work in computation of one phase or another should be done almost daily throughout the three years of the junior high school, rather than to give the whole time to it during a definite part of the course. Some advocates of certain courses in the reorganization say: "It may be

assumed that the pupil has become proficient in the four fundamental operations with integral and fractional numbers in the first six grades." Such an assumption is questionable. The teacher of junior high school grades knows, and she knows beyond the question of a doubt, that the average seventh-grade pupil is not familiar with the exact meaning of such words as sum, difference, product, quotient, factor; that the average pupil does not know how to add, subtract, multiply, or divide two small fractions; that the average pupil does not know how to write such examples with any degree of accuracy.

This is sufficient evidence that work in computation requires some very careful attention; even some very systematic attention from the very beginning of the junior high school work for a period of time determined altogether by the condition and response of the class.

Then after the class realizes the situation and understands *how* to perform the various operations it is necessary and sufficient to continue the drill for a few minutes daily. Much of this work should be oral, although at times work may be given which requires the use of pencil.

In all work in computation or arithmetic drill pupils should be made conscious of approximating results, of using short cuts, and of using constant checks. When pupils are trained to use common sense in checking their answers, many unexcusable errors will be eliminated; *e.g.*, the pupil will not say that $\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$, because he will observe that $\frac{1}{2}$ is more than $\frac{2}{6}$ or $\frac{1}{3}$; or he will not say that $102 \times 345 = 4{,}140$, because he knows that $100 \times 345 = 34{,}500$; therefore, 102×345 must be more than 34,500.

Considerable emphasis should be placed upon "short cuts"; especially where the "short cut" will save time and labor, and where it is also less likely to cause error. A few examples follow:

1. In multiplying 360 by $\frac{3}{4}$ it would be unfortunate to multiply first by 3 and then divide by 4.

2. To multiply a number by 15, multiply by 10, and then add $\frac{1}{2}$ of the product to the product; to multiply a number by 35, multiply by 30, and then add $\frac{1}{6}$ of the product to the product, etc.

3. To multiply by 25, annex two zeroes and divide by 4, etc.

When emphasis is placed upon approximating results, upon using short cuts, and upon constant checks, the abstract work really becomes thought work, and *original* thinking will be exercised by the pupil similar to that necessary in solving word problems.

The criticism is sometimes offered that the pupil grows tired and weary of so much drill work. The skillful and tactful teacher will never permit this work to become a drudgery; on the contrary, the work will be the pupil's delight. This work may be presented in such a multiplicity of ways and forms that the pupil will ask for more than the teacher can afford to give. It may be given in the nature of games; in the nature of friendly contests; in a way that the pupil appreciates the practical side of it; constant use may be made of it in the presentation of new topics and problems.

Assuming that this hypothesis has been proved, namely, that we can not afford to sacrifice facility in computation at the expense of voluminous geometric and algebraic material in the junior high school, the following suggestions are given relative to obtaining this facility:

1. Straightforward drills may be given upon the four fundamentals with integral and fractional numbers by:

(*a*) Placing tables of numbers on the board and letting pupils name results as teacher points to them.

(*b*) Using the picture of a wheel, placing numbers at the ends of the spokes, and a number in the center to be added, subtracted, multiplied or divided. The point is to see who can spin the wheel the fastest.

(*c*) Using relay races.

(*d*) Having pupils multiply a series of small numbers by a given number and then adding or subtracting another given number.

(*e*) The use of number cards.

(*f*) The use of timed exercises.

2. Applications of these abstract drills in calculating:

(*a*) Grocery bills.

(*b*) Drygoods bills.

(*c*) Personal accounts.

(*d*) Gas bills.

(*e*) Light bills.

(*f*) Pay rolls, etc.

3. Applications to problems in percentage:

(*a*) Changing per cents. to fractions and vice versa.

(*b*) In the solution of problems involving per cents. Many "short cuts" may be used in this work.

4. Formulæ in geometric work. Much stress should be placed upon substituting in formulæ, and this furnishes drill in common and decimal fractions as well as in integral numbers. The following are examples of familiar formulæ:

$$p = 2(l + w); \qquad c = 2\pi r;$$
$$A = ab; \qquad A = \pi r^2;$$
$$A = \tfrac{1}{2}ab; \qquad V = aB;$$
$$A = \tfrac{1}{2}a(b + b'); \qquad S = 4\pi r^2.$$

Pupils may be required to solve for any one of these letters, when values for the other letters are given.

5. The work of evaluating expressions may be continued still further by substituting given values for the letters in such expressions as:

1. $bc + ad$.
2. $a^2b - cd$.
3. $\dfrac{4xy + 3(x + y)}{5x - 2y}$.

6. When the pupil has reached the point where he needs drill upon collecting similar terms, coefficients that are common and decimal fractions should be used as well as integral coefficients; *e.g.*, collect:

(*a*) $\tfrac{3}{4}a + \tfrac{1}{2}b - \tfrac{1}{2}a + \tfrac{3}{8}a + \tfrac{5}{8}b - \tfrac{1}{4}b$.

(*b*) $.75c + .48d - .2c - .32d + .86d - .34c$.

7. When the pupil has learned the law of exponents in multiplication and division, many numerical examples may be given; *e.g.*,

(*a*) What is the value of $2^4 \cdot 2^2 \cdot 2^3$?

(*b*) What is the value of the expression $\dfrac{4^2 \cdot 5^3 \cdot 3^6}{4 \cdot 3^2 \cdot 5^4}$

8. In multiplication and division of monomial factors, coefficients that are common fractions and decimal fractions should frequently be used; *e.g.*,

(*a*) What is $\frac{2}{3}b \div \frac{7}{8}b$?
(*b*) What is $\frac{5}{6}a \div \frac{2}{3}a$?
(*c*) What is $.45d \times .56d$?
(*d*) What is $.21c^3 \div .32c^2$?

9. The study of special products offers an excellent opportunity for numerical applications. The following are examples:

(*a*) $41 \cdot 39 = (40 + 1)(40 - 1)$
$= 1600 - 1$
$= 1599.$

(*b*) $83 \cdot 77 = (80 + 3)(80 - 3)$
$= 6400 - 9$
$= 6391.$

(*c*) $61^2 = (60 + 1)^2$
$= 3600 + 120 + 1$
$= 3721.$

(*d*) $82^2 = (80 + 2)^2$
$= 6400 + 320 + 4$
$= 6724.$

(*e*) $79^2 = (80 - 1)^2$
$= 6400 - 160 + 1$
$= 6241.$

10. It will be of interest to the pupil to become acquainted with certain useful generalizations in addition and subtraction of fractions; *e.g.*,

(*a*) $1/b + 1/a = \frac{a + b}{ab}$; this shows how he may readily find the sum of two numerical unit fractions.

(*b*) $1/b - 1/a = \frac{a - b}{ab}$; this shows how the difference between two numerical fractions whose numerator is 1 may be found at a glance.

11. The study of quadratic equations offers further opportunity for numerical drills. When the equation is solved by completing the square, the following points are involved:

(*a*) Division involving fractional quotients.
(*b*) Dividing a fractional coefficient by 2.
(*c*) Squaring the resulting quotient.
(*d*) Adding the square of the resulting quotient to each member, and this involves addition of fractional numbers and mixed numbers.

(*f*) Further addition of integral, fractional and mixed numbers.

12. When the quadratic equation is solved by substituting in the quadratic formula, all the fundamental operations are involved, including those of raising to a power and of extracting the square root.

13. In the study of surds application may again be made with numerical expressions, both in simplifying surds and in performing all of the fundamental operations; *e.g.*,

(*a*) $\sqrt{27} = 3\sqrt{3} = 3(1.732) = 5.196.$

(*b*) $\sqrt{32} - \sqrt{8} + \sqrt{128} = 4\sqrt{2} - 2\sqrt{2} + 8\sqrt{2}$

$$= 10\sqrt{2}$$
$$= 10(1.414)$$
$$= 14.14.$$

(*c*) $5\sqrt{3} \cdot 2\sqrt{3} = 10 \cdot 3 = 30$

(*d*) $30\sqrt{5} \div 15\sqrt{5} = 2.$

14. The study of numerical trigonometry, if included in the junior high school course, offers a splendid opportunity to give the pupil a final drill on accuracy in numerical computation. It will make him especially proficient in manipulating decimal fractions, a point of great importance.

THE SLIDE RULE IN BUSINESS.

S. L. SHELLEY,

BABSON STATISTICAL ORGANIZATION.

One of the most interesting developments in the business world since the war has been the marked tendency of the average executive to desert his former methods, based on opinion and snap judgment, in favor of facts and figures. Even as Greek, Latin, and some of the modern languages are being discarded as non-essential and unnecessary in training for business, mathematics is rapidly coming into the foreground. If our experience is typical in this connection, the average executive of today places mathematics second only to English in considering an applicant's training and fitness. This tendency is indicated very clearly by the number of engineers who are being taken into executive positions which require little or no actual engineering knowledge.

The business man of today realizes as never before that all training in mathematics teaches one to think clearly and logically to a conclusion. It encourages the proving of each step in the process of reasoning before going on to the next. In other words, it teaches an accurate and scientific method which can be followed in the solution of any problem—mathematical or otherwise.

Beyond this, the business man realizes more than ever that certain phases of practical mathematics are essential in the successful handling of all executive and practically all secretarial jobs.

It is some of these practical phases—of definite value in the business world—that I want to discuss.

In keeping with the tendency I have just outlined, it is probable that the business man of today is doing twice as much actual figuring as he did before the war. In the old days he had a general idea about selling costs, his overhead and his profit; today he demands exact figures in detail for every step in his business transaction from the purchase of raw materials,

through the process of manufacturing, and out into his channels of distribution. He must not only have the definite figures, but he must have those interpreted into percentages of increase and decrease for comparison with previous records, quotas and carefully figured estimates.

While the calculating machine has shouldered a large part of this additional work, there is still a portion of it to be done by the individual. As far as I know, the only efficient tool for handling this personal calculating is the ordinary slide rule. We find scores of university trained men using it at their desks daily and considering it as much a part of their equipment as the fountain pen and Eversharp pencil. There are probably ten times as many executives who have use for it, but who are not employing it because they don't know anything about it.

To be sure the slide rule is not accurate beyond three or possibly four figures, but a large part of its personal work is figuring percentages, which can be handled very easily by it. If I may quote a personal experience in this connection, I may say that I taught my secretary to use the slide rule last spring. In July, when making up the annual report covering some 2,500 insertions of advertising, she made over 10,000 separate calculations on it in two and one-half days. This same multiplication and division would have taken about three weeks long hand or about a week on the calculating machine.

While we are talking about practical mathematics in business, I am going to take the liberty of calling your attention to another instance in which it is playing a part of ever-increasing importance. The business man is not only demanding facts in place of guess work, but he is demanding his facts in picture form. Some one has said that a picture is worth a million words; certainly a good chart is worth several pages of written report in getting and holding the attention of the average executive.

The properly trained assistant of today must not only be able to get the figures, but must be able to picture them so that they will be understood instantly and so that they can not be misunderstood.

We find that as a general rule the business man is not even

as well informed on charting as he is on efficient calculation. Of our 16,000 clients who probably represent the most progressive group of executives in the country, we found that the majority were using ordinary square rule charts to picture all sorts of facts and figures. These, of course, are perfectly accurate and truthful in measuring units, but when it comes to depicting rate of growth or percentages they lie most brazenly. Because they see it in a chart in black and white, the average man is ready to accept its indication without question.

Percentages and rates of growth should always be shown on ratio or logarithmic charts which will present such material correctly and indicate an accurate conclusion.

We would like to see both the slide rule and charting taught as early in a course of mathematics as possible. Personally, I don't believe that it is necessary to wait until the student is advanced far enough to understand the theory of logarithms; in fact, we have several secretaries in our own office who are using both the slide rule and ratio chart accurately and efficiently, who have no idea of the theory on which they are based. I urge most strongly, in any event, that when charting is taught at all, that both types of charts be explained; otherwise, the student is liable to get the impression that the ordinary square rule chart is sufficient for all work. He should at least understand that there is another form which should be used under certain circumstances.

If I were to interpret for you the attitude of the average business man toward mathematics today, I believe that it would be something as follows: "Teach as much mathematics as you can; the more, the better. Get the following practical things into your course as early as you possibly can, certainly before the student has finished high school: (1) Teach him to add; (2) teach him to subtract; (3) teach him to multiply; (4) teach him to divide; (5) teach him simple equations; (6) teach him measurement of all kinds; (7) teach him charting, so that he can both understand and make charts; (8) teach him the slide rule, that he may save his own time and mine in the numberless calculations that come up in the transaction of business."

A PROGRAM OF INVESTIGATION AND COOPERATIVE EXPERIMENTATION IN THE MATHEMATICS OF THE SEVENTH, EIGHTH AND NINTH SCHOOL YEARS.

RALEIGH SCHORLING AND JOHN R. CLARK,

THE LINCOLN SCHOOL OF TEACHERS' COLLEGE.

The Present Status of Mathematics in Grades Seven, Eight, and Nine.—There appears to be widespread dissatisfaction with the content and the organization of traditional materials of instruction in these grades. In particular, the work of these years is characterized by waste of time and effort in the teaching of insignificant subject matter. There is little defense against the charge that the conventional seventh and eighth grade mathematics devotes much time to unmotivated and unproductive reviews; that much of the social-economic arithmetic (taxes, insurance, stocks, bonds, and the like) is beyond the comprehension of the pupils of these earlier grades; that common sense estimation in solving problems does not receive adequate exercise; and that American practice has not begun to profit by the experiences of European schools in giving to pupils of these grades an outlook and an appreciation of the significance and meaning of mathematics beyond arithmetic.

Even less defense is offered for the conventional algebra of the ninth school year. Conservative mathematicians have recognized the need of reconstruction in this grade. Extreme emphasis upon the manipulation of symbolism, which to the pupil does not have meaning and significance, together with poorly selected content and material of little value even in later courses of mathematics, constitute the two pronounced indictments of the standard algebra of the ninth grade. Progressive teachers of mathematics and many school administrators are demanding a reconsideration of the content and methods of teaching in this grade.

Variability in Recent Junior High School Courses in Mathematics.—The strongest evidence of the fact that the mathe-

matics problem of these grades has not been satisfactorily solved is found in the marked variation in the existing series of texts for these years. In the content and grade placement of various books we have clear evidence of distinctly different hypotheses concerning the curriculum for these grades. One writer, for example, holds that the seventh grade should be given entirely to arithmetic and the eighth and ninth grades to general mathematics. Another advocates geometry as the major work of the seventh grade, arithmetic for the eighth, and algebra for the ninth. The usual practice is based upon the theory that little geometry and less algebra should be given prior to the ninth grade.

Scarcely any critic would argue that these alternative schemes are equally effective in forming desirable habits, giving useful information and increasing mathematical power. Experimentation involving a very careful testing of methods and results should eventually disclose the most effective use and organization of material. It is urgent that students of education evaluate these courses.

Bases for the Selection of Subject Matter.—The validity of proposed subject matter will be determined by the degree to which it serves worthy individual and social needs and to which it meets the requirements of a sound psychology of learning. When the materials of instruction have been fairly well determined, the arrangement (grade placement and sequence of details) becomes a psychological problem, the solution of which will depend upon critical analyses of children's responses to these materials.

Conventional curriculum practice proceeds in quite the opposite manner. The tendency to continue existing practice has kept the curriculum more or less static. A careful student of education a generation ago might have suggested the approximate type of reorganization in mathematics which is being affected today. In fact, Professor Hanus, of Harvard, in his addresses to the Harvard Teachers' Association twenty-eight years ago, advocated a new curriculum including many of the reforms now receiving attention.

Desirability of Organizing Subject Matter in Accordance with Known Principles of Learning.—Modern psychology of

learning states that children learn the concepts and principles of mathematics as they learn other meanings—through abundant, varied and purposeful experience. In the junior high school grades numerous attempts to improve the curriculum take the form of "shoving down" the mathematics from the senior high school grades. Much of the present dissatisfaction is due to the fact that these courses were shifted downward without an adjustment to the needs, capacities and interests of less mature minds. This temptation is particularly noticeable in those communities which are instituting junior high schools without a clear philosophy of the junior high school curriculum and with teachers who are held by traditional practice. The new movement clearly requires that a new spirit and method shall be developed in these grades. The details of this technique can be discovered and arranged only on a basis of extensive classroom studies.

Knowledge of the Details of Learning in Mathematics Vague.—The evidence that our knowledge of the details of learning in mathematics is inadequate is seen in the great number of debatable issues which can be raised. For example, assuming agreement upon the content of the course, it may be asked:

(*a*) Should the materials be presented under the headings of arithmetic, algebra and geometry, or should these materials be merged into general mathematics?

(*b*) To what extent should the topical organization of this course persist? To what extent may the organizing principles of the course be found within the subject itself as contrasted with the possibility of presenting a series of projects designed to teach the necessary skills and information?

(*c*) Would children incidentally (without classroom instruction) acquire the knowledge of the minimum essentials of the subject? In other words, does classroom teaching result in consuming time with material which would be learned anyway?

(*d*) What constitutes sufficient practice for the necessary skills?

(*e*) What is the most desirable frequency and length of review periods?

(*f*) How much practice, at the time of initial teaching, should a new topic receive?

(*g*) What specific skills, information and knowledge of principles do children possess when they come to the seventh grade?

(*h*) Is there a definite relation between speed and accuracy in computation? Does increasing either increase the other?

These and other questions which occur to any critical teacher can be answered only after extensive studies of children's responses.

General Principles of Agreement.—Exponents of the reorganization movement in mathematics appear to be in agreement upon the following considerations:

1. The threefold purposes of teaching junior high school mathematics is: (*a*) to make school studies and life out of school mean more to a boy or a girl than they otherwise would; (*b*) to induce the pupil to react to a variety of human activities, which yield greater satisfaction; and (*c*) to give to the pupil more ready and accurate control of the numerical and spatial relations of human life.

2. The course in each year should give the pupil the most intrinsically valuable mathematical information and training which he is capable of receiving at that time, with little consideration for the needs of subsequent courses.

3. The content of these courses must be built from the point of view of the children, and from the consideration of social needs, and not solely from the logical requirement of mathematics.

4. The general aim stated at the outset will necessitate the inclusion of certain elements of arithmetic, intuitive geometry, algebra, numerical trigonometry and statistics.

5. Throughout the courses the idea of relationship or of the dependence of one quantity upon the other is to be emphasized. From the mathematical point of view, this notion of function is the unifying principle; but from the point of view of teaching, the basic, guiding principle is not found within the science itself, but within the children's learning.

6. In organizing the course the usual emphasis upon the special divisions of mathematics and the customary time allot-

ment should be replaced by the introduction of the topics in such a way as to insure a maximum of direct and intensive application, flexibility and significant interrelations.

7. The mathematics of the junior high school should mark the end of the required mathematics, and hence it must throw into bold relief those general mathematical ideals, tools and habits which are now regarded as of maximum importance.

8. The unity of space and number should persist throughout, for geometry furnishes a concrete source, setting and illustration of significant number relations. In consequence, measurement is one of the fundamental processes by which the pupil may discover the significant number relationships directly through the senses.

9. Manipulation as an end is to be eliminated. Mechanical work can be justified only when necessary for understanding and using fundamental principles.

10. Attention should be directed toward a better appreciation of the part that mathematics has played and is playing in the progress of civilization, in order to give a better understanding of common situations.

11. The material should be vitalized through the early introduction of principles that are commonly delayed until the later courses, as, for example, numerical trigonometry; and through a closer correlation with other school subjects, as, for example, elementary science, mechanics, industrial art, fine art.

12. The material is to be socialized through the extension of units of instruction from classroom exercises and topics through a series of activities, projects, or problems requiring (1) coöperation and (2) sharing of interests, efforts and results. The course should aim to give rigorous discipline in things worth while. The ability of children to undertake and carry through purposeful activities should be recognized. The course should capitalize the rigorous discipline that comes in engaging and carrying through purposeful activities, instead of trusting to the discipline of activities which are often uninteresting ends in themselves.

13. Throughout junior high school mathematics common sense in computing with approximate data should be exercised.

14. Through observation, measurement, intuition and a con-

sideration of elementary properties of geometrical figures, the course should lead to control of symbols, vocabulary, and the principles of space relations which common experience requires.

15. Junior high school mathematics should teach the necessary social and economic uses of arithmetic. The more technical forms of business practice, such as insurance, brokerage, stocks, bonds, etc., should be placed late in the course to utilize the greater maturity, experience and mathematical knowledge of the pupil.

16. A marked increase in the accuracy of computation with integers, fractions (common and decimal) and per cents is imperative in these grades.

The tendency toward agreement upon these general principles is due chiefly to the National Committee on Mathematical Requirements and to its many coöperating organizations. The very rapid progress which has been achieved in the short period of the activities of the National Committee is an illustration of the effectiveness of a coöperative enterprise undertaken by a carefully selected group of teachers. If the curriculum is to be rapidly improved, it is highly desirable that the progressive workers in this field continue to coöperate.

Two Types of Investigations.—The preceding general principles constitute the point of departure for the more detailed investigations which are necessary to translate this program into schoolroom practice. Two types of investigation appear to be involved: (1) a series of researches and investigations designed to determine a body of curriculum material which can be defended on a basis of social worth. These are for the most part individual in character; (2) a series of classroom studies (psychological studies) calling for coöperative efforts which have been referred to in the preceding section, and which will be discussed in detail later.

Studies in Content.—The writers have initiated, with the intention of completing within a short time, the following studies, bearing on the determination of a valid content for these grades:

It may be helpful to examine briefly each of the steps in this program in order to see the kind of study which is under way.

I. What does the first six years of arithmetical instruction

achieve? The existing tests show fairly well the ability of children to compute with integers and fractions, and they tell us something about their ability to reason. But we have little information concerning the knowledge of symbols, processes, terms, principles and relations which sixth-grade children may possess.

In consequence, an inventory test has been developed. This, together with the compilation of results from widely used existing tests, will yield a reasonably complete account of mathematical instruction at the close of the sixth grade.

II. A social survey (industrial, commercial, and professional) to determine the mathematical materials that can be justified on basis of their social usage.

This is an investigation of the quantitative aspects of economic and social needs. Certain efforts to inventory social needs have been made in several doctors' theses, as, for example, the study by Wilson, but the technique used in these studies is challenged on two counts: (1) the data were secured in devious ways, and (2) it may be questioned whether discovering "what is" is a valuable index to "what should be." An inventory of various citizen groups showing that they think inaccurately about things mathematical in life, and that they dodge efficient methods in treating things quantitatively, does not prove that people should not be trained to think clearly and accurately about mathematical things. For example, the fact that relatively few people in the United States know the metric system is not sufficient evidence for concluding that the whole nation, or indeed the whole world, should not be trained to use a standard metric system.

This part of the investigation is very difficult, but its importance is great. We can get some help in finding out what the general reader needs by a quantitative study of the frequency of occurrence of mathematical symbols, terminology and facts in Sunday newspapers, popular magazines, advertisements, articles, and the like. Probably the solution of this problem lies in bringing together a sufficient number of job analyses from which specific common elements may be selected.

III. What material should be taught in these grades? The materials which have been assembled by the survey should be

evaluated in the light of certain considerations other than frequency of use. For example, the survey may show that relatively few people know the marks of a good investment, or know how to compute compound interest, yet both of these elements might be included in the curriculum because of other considerations. There are ten or more of these considerations before which the elements of the new curriculum must "pass in review." Of these five are given as illustrations.

1. Complete analysis of texts.
   (*a*) Arithmetic for grades 6–8, inclusive;
   (*b*) At least two or three French and German texts covering grades 6–9;
   (*c*) Junior high school texts.

This analysis of texts will result in a series of charts, one showing the material that was taught in the arithmetics ten years ago, a second one showing what recently published texts include, and a third the material that is contained in the junior high school texts. The practice of our public school is, of course, determined almost wholly by the material and organization of text-books. A comprehensive survey in graphical form of the material that is now in use raises many questions, as, for example, relative emphasis, reasons why certain topics appear and disappear. Obviously such a study facilitates a critical review of the wide range of proposed materials. It does not determine what ought to be taught, nor will it be weighted heavily.

2. An analysis of materials used in schools undertaking innovations in mathematics.

There are about 175 schools in the country that are willing to be called experimental schools. Of this number, probably about 15 believe that they have put forth fruitful efforts in reorganizing mathematics. However, the work of these 15 schools is not experimental in a scientific sense—that is, there is little careful study of a precisely defined problem under controlled conditions. At best, this experimental work in mathematics takes the form of innovations. These innovations are undertaken by a group of teachers unusually well prepared in mathematics, both in content and in professional

courses. In consequence, the composite opinion of this limited group of teachers is a most significant guide to those who desire to improve the teaching of junior high school mathematics.

3. A collation of all committee reports on the mathematics of the junior high school grades.

Writers of text-books have in turn been affected by committee reports. Hence it is essential to make a collation of committee reports to see to what extent the elements in the different reports have persisted. Committee reports at their best represent the composite opinion of competent teachers. The significant report on junior high school mathematics recently issued by the National Committee on Mathematical Requirements is of unusual importance.

4. A survey of the literature, articles and studies which bear on the problem.

The material listed above needs to be studied for the opinions of those who have given especial thought to this problem. The literature of high school mathematics has a number of very thoughtful papers with which teachers are not generally familiar. These papers need to be collected and studied to see whether practice can be improved by the counsel which this literature includes.

5. Mathematical needs of other subjects.

This takes the form of a preliminary survey of junior high school mathematics to see to what extent it contributes toward making pupils intelligent in other school subjects. We need to know what materials in other school subjects can be brought into the mathematics classroom and used to illustrate mathematical principles, and we need to know to what extent mathematics can be used in other school subjects. The question of transfer to other subjects will not be so important as the question of direct use in making these subjects more meaningful.

Steps IV, V, and VI will be discussed in the next section under "Studies in Learning."

A Program of Coöperative Research.

Studies of Learning.—It has been stated that one of the methods to be used in developing a valid curriculum is that of

coöperative experimentation. The study of children's responses to proposed materials of instructions must be made by many teachers with many types of children and under a variety of school conditions.

A group of teachers, including the writers of this article, are engaged in making detailed classroom studies of this character. In general, this program takes the form of choosing one or more seventh-grade sections for intensive study. The detailed steps in the program include the following:

1. The taking of a complete inventory of the mathematical equipment of beginning seventh-grade pupils. One phase of this inventory refers to the first investigation listed in the preceding section. This test consists of three parts. Each part can be completed by all pupils in the thirty-minute period. The remainder of the inventory consists of a compilation from existing standard tests. Any teacher who makes a special study of mathematics with seventh-grade children is primarily interested in the mathematical growth of her children, but no valid conclusions can be achieved without complete tests before and after the subject is studied. In consequence, this inventory test has been designed to be given at the beginning of seventh-grade work.

2. The teaching of at least two distinct types of instructional material, one of which is constructed especially for experimental purposes. The other may be chosen from one of the junior high school series that is now available, or it may be any standard arithmetic in use in the particular school system. It should be noted that the many types of material now available indicate distinct hypotheses concerning the selection and placement of material in these grades. One outcome of this experiment will be a comparison, under controlled conditions, of the effectiveness of these hypotheses.

3. A critical observation of the children's responses to the type of material taught. A record will be kept of pupils' successes, failures, interests, the sufficiency of explanation, the adequacy of practice material, the clearness of terminology, the degree to which the subject matter is taken from the experiences of the children and the suitability of the projects which are included. Special blanks will be provided for these rec-

ords in order that the teacher may not find this technique an added burden.

4. A systematic use of measurement to test the pupil's mastery of each unit of material and growth in skill, information and mathematical power. In this, the order being teaching, testing, and further teaching by means of practice material especially selected to supplement instances of inadequate learning, and finally a comprehensive inventory test of the outcomes of teaching each year's work. From these inventories comparative data on the effectiveness of the two types of material will be secured.

5. A similar intensive study for grades 8 and 9. The printed materials used by seventh-grade pupils are now available for use. This material is a printed pamphlet which can be secured from the writers of this article.

An Outline of the Program as a Whole.

The two types of investigations will be used in carrying out the following program:

I. What mathematics do children know when they come to the seventh grade?

II. A social and economic survey to determine the mathematics materials that can be justified upon the basis of social usage.

III. The evaluation of possible materials of instruction for grades 7, 8, 9, resulting in a tentative curriculum of what ought to be taught.

IV. The organization or arrangement of these materials in accordance with known principles of learning.

V. The teaching and *testing* of materials to ascertain the degree to which the objectives have been attained.

VI. A final revision of the materials to embody any needed changes of order, of emphasis, or of method which may have been suggested by the teaching-testing step.

A considerable number of schools will participate in this coöperative program. It is, of course, desirable that representative types of pupils and teaching conditions be included. In particular, it is desirable to have the constructive help of superintendents and teachers who have a keen interest in im-

proving the effectiveness of instruction in these grades. The number of schools which can be added to this list must necessarily be limited in order that the experiment may not become unwieldy. Hence additional selections of schools will be made on evidence of particular interest in the problem.

Materials to be Used by the Coöperative Schools.*

The following types of material are included in the program: (1) the inventory tests, (2) experimental teaching material for seventh-grade pupils, (3) test and record blanks which will economize the time of the teacher and assist in helping to analyze pupils' learning. This material may be obtained from the writers.

The members of this and other coöperative groups have found satisfaction and profit in the larger professional vision and keener critical interest in an intensive study of a well-defined series of problems. Of the various problems of mathematics, the one fundamental at the present moment is surely that which is concerned with collecting the proper materials of instruction and determining their proper sequence and grade placement. This proposed experiment offers an opportunity to mobilize the efforts of those who are thoughtful and critical about what mathematics children should learn, and how it can best be learned.

* These materials will be supplied at approximately 80 cents per pupil in class lots. Individuals may secure copies of the materials for the seventh grade at $1.00 per copy.

TESTING AS A MEANS OF IMPROVING THE TEACHING OF HIGH SCHOOL MATHEMATICS.

E. R. BRESLICH,

UNIVERSITY OF CHICAGO.

Testing, in the sense in which the word used in this article does not refer to the so-called standardized test, but to the ordinary class examination, or class test.

In a modern business every detail in the manufacture of an object is examined critically. Similarly, the teacher should employ a critical examination to measure the results of his teaching. Tests may be used to show the complexity of the work and to reveal definite problems for the teacher. Indeed, it is found that they present more problems than they solve. It may be said that the principal value and aim of all testing should be to improve the quality of teaching by means of an analytical exposition of its problems and difficulties. Opinion must be replaced by knowledge, guess work by evidence. This type of testing is fundamental in the progressive organization of a course.

Briefly stated the method might be as follows: First, some concrete aims are set up and agreed upon. Second, tests are designed and given, to establish to what extent the abilities to be developed in the pupil have been attained. Third methods of instruction are developed that are adapted to improve these abilities. Finally, suitable tests are devised to determine whether the course is actually helping the pupil to gain in these abilities in order to help him further if he is still found deficient.

This seems a very simple and sensible program. Following this program we shall first make in mathematics a list of the important concrete aims of the chapters, principles, or topics taught, such as the development of the abilities to manipulate the formal operations of algebra, to translate verbal problems into algebraic symbols, to grasp space relations, to classify and to generalize. Having selected our subject matter ac-

cording to our best judgment, the next step will be the designing of tests by which to measure the achievement of the pupils with this subject matter. The problem of constructing such tests is as yet only in the beginning stage.

That the ordinary class examination does not always establish the facts desired may be seen from the graphs in Fig. 1. They represent the grades of 50 pupils in a test given to three classes taught by three different teachers in a third-year course. Although classes and teachers vary in ability, the general shape of the graphs is about the same. They show that either the pupils do very well in the test, or they fail. It is hardly possible that in all classes the pupils were either very poor or very good, and we may assume that the teachers were at least better than the average. Evidently, the test gives no information about the poorer students, for the work seemed to be so hard that they were unable to show what they could do. Nor does it tell anything about the bright students who were able to

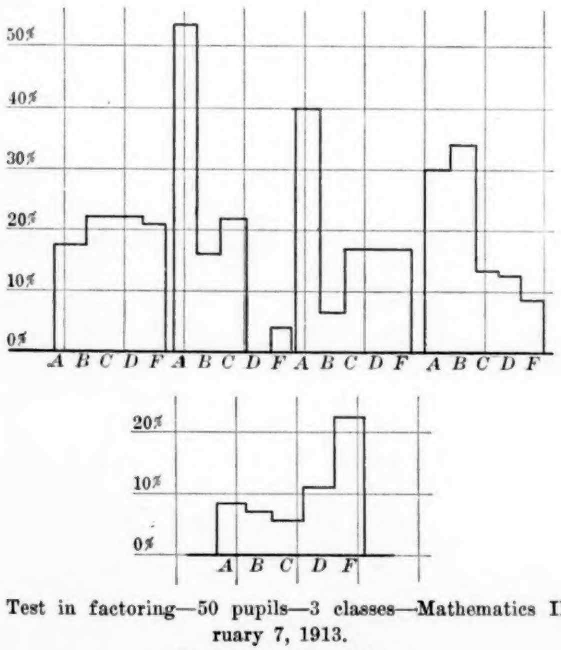


FIG. 1. Test in factoring—50 pupils—3 classes—Mathematics III., February 7, 1913.
Classes I., II. and III.

complete the test in less than the required time and therefore could not show what they were really able to accomplish. A more suitable test must contain enough work of the simple type for the slow pupil. It must also contain some difficult work which only some of the best pupils can do. The graphs in Fig. 2 show that 13 per cent. of the pupils failed. Apparently

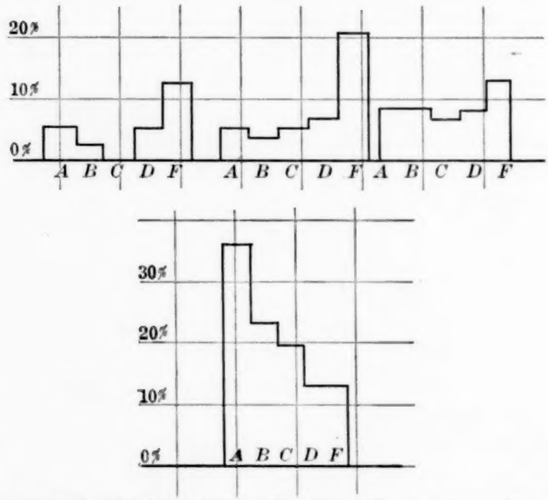


FIG. 2. Test V. Mathematics I., Chapters VI. and VII. Four classes.

in this test the slow pupils were able to show that they could do some of the work.

However, in this as well as in the first test, a large percentage of pupils failed to gain real mastery of the work in which they were being tested, and needed further teaching. A more detailed study must be made to reveal what the nature and extent of this instruction should be.

To measure the results in a test each problem is divided into the component elements which enter in the solution. Each problem is analyzed as to the processes it aims to test. A score of 1 is then given for each process performed correctly. If not correct it is scored 0. Attainment it the ratio of the actual score made to the largest possible score. Thus, if a process is performed correctly by 76 pupils in a group of 80, the ratio $76/80 = .95$ is the attainment.

Before the test is given, a key for scoring the various processes is worked out. The following are a few examples showing how this may be done for some of the typical work in algebra and geometry.

OPERATIONS.

1. Multiplication of monomials is one of the simplest problems in algebra. It is also one of the most fundamental processes, and must therefore be mastered by all pupils. The problem involves three steps, the determination of the sign, the arithmetical product, and the literal product.

Each of these is scored separately. For example, in $(-12m^5)(-16mr^3)$,

$$\left.\begin{array}{ll}\text{the sign is } + \ldots\ldots\ldots\ldots & 1\\ \text{the arithmetical product is } 192 \ldots\ldots & 1\\ \text{and the literal product is } m^6r^3 \ldots\ldots & 1\end{array}\right\}\begin{array}{l}\text{Therefore,}\\ \text{the score}\\ \text{is 3.}\end{array}$$

It must be remembered that the score does not indicate a measure of the degree of difficulty or a value of the step.

2. The *division of monomials* involves the same three steps. Thus in $-6a^2cx^2y \div 2axy$, the sign

$$\left.\begin{array}{ll}\text{is } - \ldots\ldots\ldots\ldots & 1\\ \text{the arithmetical quotient is } 3 \ldots\ldots & 1\\ \text{the literal quotient is } acx \ldots\ldots & 1\end{array}\right\}\begin{array}{l}\text{Therefore,}\\ \text{the score}\\ \text{is 3.}\end{array}$$

3. The component parts of *multiplication of a polynomial by a monomial* are the multiplications of a number of monomials. In the following example this occurs three times. Therefore the score is 3, meaning that three operations have been performed correctly:

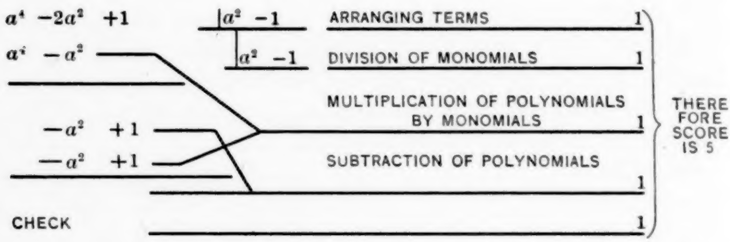
$$\left.\begin{array}{rl}(3a^3-2a^2+5a)(-2a^2) = -6a^5 \ldots\ldots & 1\\ +4a^4 \ldots\ldots & 1\\ -10a^3 \ldots\ldots & 1\end{array}\right\}$$

4. *In the multiplication of polynomials* we have the multiplication of a polynomial by a monomial, and the addition of the resulting polynomials.

Accordingly, the product $(a^3-a^2-1)(a+1)$ is scored as follows:

$$\left.\begin{array}{ll} a^4 - a^3 - a \ \ldots\ldots\ldots\ldots\ldots\ldots & 1 \\ \quad + a^3 - a^2 - 1 \ \ldots\ldots\ldots\ldots\ldots & 1 \\ \hline a^4 \qquad - a^2 - a - 1 \ \ldots\ldots\ldots\ldots & 1 \end{array}\right\} \text{Therefore, the score is 3.}$$

5. *In the division of polynomials* we have the following steps.



Similarly we may now show how to analyze the solution of equations.

EQUATIONS.

6. *Solution of a linear equation:*

$$y - \frac{2y-4}{3} = \frac{2(5-y)+7}{9}.$$

The processes are

1. Clearing of fractions:

   $9y - 3(2y-4) = 2(5-y) + 7$ 1
2. Multiplying polynomials by monomials:

   $9y - 6y + 12 = 10 - 2y + 7$ 1
3. Combining similar terms:

   $3y + 12 = 17 - 2y$ 1
4. Adding to, and subtracting from both members:

   $5y = 5$ 1
5. Dividing both members:

   $y = 1$ 1
6. Checking in the original equation:

   $\left.\begin{cases} 1 + 2/3 = (8+7)/9 \ \ldots\ldots\ldots\ldots \\ 1\,2/3 = 1\,2/3 \ \ldots\ldots\ldots\ldots \end{cases}\right\} 1$

Therefore, the score is 6.

7. *Solution of a quadratic equation.*

$4x^2 + x - 39 = 0$

$$\left.\begin{array}{lr}
(4x+13)\ (x-3)=0 \text{ two factors} \dots\dots\dots\dots & 2 \\
\left.\begin{array}{l} 4x+13=0 \\ x-3=0 \end{array}\right\} \text{two linear equations deducted} \dots\dots & 2 \\
\left.\begin{array}{l} x=-13/4 \\ x=3 \end{array}\right\} \text{two equations solved} \dots\dots\dots\dots & 2
\end{array}\right\} \begin{array}{l}\text{Therefore,} \\ \text{the score} \\ \text{is 6.}\end{array}$$

These scores are used when the process of solving the equation is first taught. When these equations occur later the complete solution may be considered as one operation.

8. *Simultaneous equations.*

(*a*) Solution by graph:

$$\left.\begin{array}{lr}
\text{1. Make tables} \dots\dots\dots\dots\dots\dots & 2 \\
\text{2. Draw axes, and select units} \dots\dots\dots & 2 \\
\text{3. Plot points, and graph lines} \dots\dots\dots & 2 \\
\text{4. Solve by finding point of intersection} \dots & 1
\end{array}\right\} 7$$

(*b*) Solution by addition or subtraction:

$$\left.\begin{array}{lr}
3m+7n=34 & \\
7m+8n=46 \quad \text{multiply} & \\
21m+49n=238 \dots\dots\dots\dots\dots\dots & 1 \\
21m+24m=138 \quad \text{subtract} \dots\dots\dots\dots & 1 \\
25n=100 \dots\dots\dots\dots\dots\dots & 1 \\
n=4 \quad \text{solve} \dots\dots\dots\dots\dots & 1 \\
3m+28=34 \quad \text{substitute} \dots\dots\dots\dots & 1 \\
\left.\begin{array}{l} 3m=6 \\ m=2 \end{array}\right\} \text{solve} \dots\dots\dots\dots\dots\dots & 1 \\
(m,n)=(2,4) \quad \text{results} \dots\dots\dots\dots\dots & 1
\end{array}\right\} \begin{array}{l}\text{Therefore,} \\ \text{the score} \\ \text{is 7.}\end{array}$$

VERBAL PROBLEMS.

The verbal problems of algebra are subdivided into a similar scheme.

9. *Motion problem.*

A courier who travels 6 miles an hour is followed after 2 hours by a second courier who travels $7\frac{1}{2}$ miles an hour. In how many hours will the second courier overtake the first. The following processes are involved.

1) Method:
2) Data for one courier:
3) Data for the second courier:

t	r	d	
x	6	$6x$	1
$x-2$	$7\frac{1}{2}$	$7\frac{1}{2}(x-2)$	1

(Method . . . 1)

4) Stating the equation: $6x = 7\frac{1}{2}(x-2)$ 1

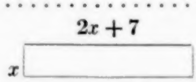
5) Solving the equation: $\left.\begin{aligned} 6x &= \frac{15x-30}{2} \\ x &= 10 \end{aligned}\right\}$ 1

The score for the whole problem is 5.

10. *Geometric problem solved algebraically.*

One side of a rectangle is 7 feet longer than twice the other, the perimeter being 54 feet. Make a sketch of the rectangle and find the length of the sides.

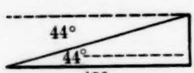
1. Let x be the number of feet in one side
   Then $2x$ $7+$ is the numbor of feet in the other side . . . . data 1
   Figure:



2. State equation: $2x + 4x + 14 = 54$ 1
3. Combine: $6x + 14 = 54$ . . . .
4. Subtract: $6x = 40$ . . . . solution . . 1
5. Divide: $x = 6\,2/3$ . . . .
6. Other solutions: $2x + 7 = 20\,1/3$ 1
7. Check: $6\,2/3 + 6\,2/3 + 20\,1/3 + 20\,1/3 = 54$ . . . 1

Therefore the score is 5.

11. *Problems In Indirect Measurement.*



x Figure and notation 1
Alternate int. angles are equal 1
$\tan 44° = x/450$ Choice of function 1
$x = 450 \tan 44$ Solution 1
$x = 450\,(.966)$ Use of table 1
$x = 434.7$ Arithmetical computation 1

Therefore, the score is 6.

Geometric Problems and Theorems.

In the scoring of *theorems* or *problems* in geometry the following general method is used:

1. Figure and helping lines	1
2. Given, to prove	1
3. Explain how helping lines are drawn	1
4. Congruent triangles	4
5. A single statement giving a correct inference with the correct reason for it	1

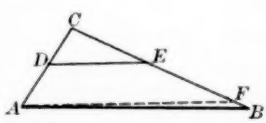
The following proof illustrates the method.

12. *Use of the indirect method of proof.*

Theorem: Two lines that cut two intersecting lines and make the corresponding segments of the given lines proportional are parallel.

Given: DE and AB cut by CA and CB, making $CD/DA = CE/EB$ ......
To prove: $DE \parallel$ to AB ...... } 1

Figure:

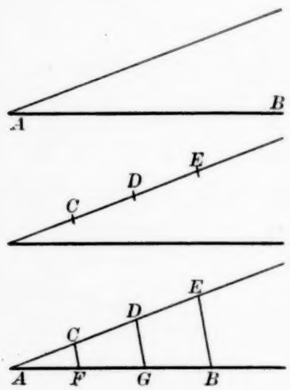


Proof:	Assume DE not parallel to AB	1
	Draw AF parallel to DE	1
	Then $CD/CA = CE/EF$	1
	But $CD/DA = CE/EB$	1
	$\therefore CE/EB = CE/EF$	1
	$\therefore CE \cdot EF = CE \cdot EB$	1
	$\therefore EF = EB$	1
	This is impossible	1
	$\therefore$ The assumption is wrong	1
	CF parallel DG parallel EB	1

13. *A construction with ruler and compass.*
To divide a segment into three equal parts.

Given: A segment AB
Required: To divide AB into 3 equal parts } 1

Construction:



Draw helping line.......... 1
Lay off 3 equal segments... 1
Draw parallels 1

Proof: $AC = CD = ED$ 1
CF parallel DG parallel EB 1
$AF = FG = GB$ 1

To obtain a clear understanding of the study of tests we may examine in detail the results of one of the tests given to a class in First-Year Mathematics. None of the problems in this test were taken from the text book used by the pupils. The test is as follows.

Test VIII., Mathematics I.

I. Change to the simplest form by collecting terms

$$6x^2 + 8 - 3x^2 - 5 - 9x^2 - 7$$

$$3g + f - 8g - 6f - 7g - 5f$$

$$-8t^3 - 4t + 6t^2 - 3t + 2t^3 = 9t$$

$$-2ab - 3g + 7ab + 7g + 6ab \div 4g$$

II. If $c = 4$ and $f = 2$ what does $2c^3 - 3f$ equal?
if $a = 3$ and $b = 2$ what does $3ab + ab^2$ equal?
if $x = 3$ and $y = 4$ what does $xy^2 - 2xy$ equal?
if $r = 2$ and $s = 4$ what does $r^2 + 3r^2s$ equal?

III. From $2d + 13f$ take $7d + 14f - 6g$

Take $9h + 14k$ from $7h - 3k + 8s$

Subtract $6m - 11n + 13p$ from $m + 3n + 7p$

IV. Perform the indicated operations and reduce to the simplest form:

$3(7y) =$ $\qquad$ $(4x)(-3xy^3) =$

$(2a)(4ab^2) =$ $\qquad$ $(a^3)(-3a)(-2a) =$

$2/3$ of $9m =$ $\qquad$ $(-3xy^3)\,4 =$

V. Simplify $7(5x + 9)$; $5(8x - 4)$; $-6(5x + 2)$; $-9(4x - 6)$; $7(-6x - 4)$; $-8(-4x - 6)$.

VI. Perform the indicated operations and reduce to the simplest form:

$$\frac{12n}{4} = \qquad \frac{7a}{15} \div \frac{7a^2}{20} =$$

$$6c^3 \div 2c^2 = \qquad \frac{-12x^2y^2 \cdot (x-2)}{-3x^2y^2} =$$

$$\frac{-8a^2b}{4a^2} =$$

$$\frac{4x^4}{5} \div 2x^2 =$$

VII. Perform the indicated operations and reduce to the simples form:

$$(2a^2 + 7a - 9) \quad (5a - 1) =$$

$$\frac{18m^2n - 27mn^2}{9mn} =$$

$$(x^3 - x^2 - 4x + 4) \div (x^2 - 3x + 2)$$

VIII. Show graphically each of the following statements to be true. Explain your drawing.

$$(+8) + (-2) = +6; \ (-6) - (-2) = -4;$$

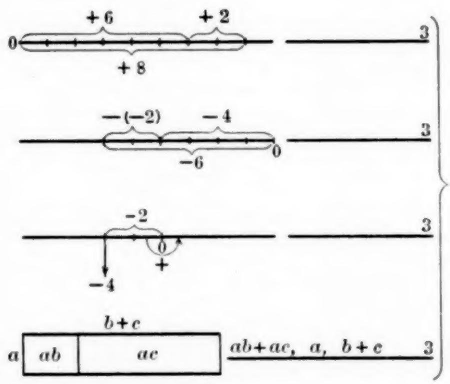
$$(-2) \quad (-4) = +8; \ ab + ac = a(b + c)$$

The first step in the study is to make the key for scoring. To avoid needless repetition only part of the key used in scoring the test above is shown as follows.

Key for Test VIII., Mathematics I.

I.	$-6x^2-4$ Combining terms	2	Total score 9.
II.	$2 \cdot 16-24=+8$ Substitute, multiply, combine	3	Total score: 12.
III.	$-5d-f+6g$ Each term is scored 1, if right	3	Total score 9.
IV.	$21y$ Numerical product	1	Total score: 13.
	$8a^2b^2$ Numerical product, literal product.	2	
	$6m$ Numerical product	1	
	$-12x^2y^2$ Sign, numerical product, literal product	3	
	$+6a^5$	3	
	$+81x^4y^{12}$	3	
V.	$35x+63$. Each term scores 1, if correct	2	Total score: 12.
VI.	$3n$ Arithmetical quotient	1	Total score: 15.
	$3c$ Numerical part, literal part	2	
	$-2b$ Sign, numerical part, literal part	3	
	$2/5\,x^2$ Invert, multiply, reduce	3	
	$4/3a$ Invert, multiply, reduce	3	
	$+4(x-2)$ Sign, numerical part, literal part	3	
VII.	$10a^3+35a^2-45a+$ Polynomial by monominal; $-2a^2-7a+9$ Polynomial by monomial; $10a^3+33a^2-52a+9$ *Combine*	3	Total score: 8.
	$\dfrac{9mn\,(2m-3n)}{9mn}$, $2m-3n$ Factor, reduce	2	
	x^3-3x^2+2x, $2x^2-6x+4$, $x+2$ Divide, multiply, subtract	3	

VIII.
Total
score: 12.



It appears that ninety different operations were to be performed in this test.

Having scored each process of every problem for every paper by following the directions of the key, we find for every process the ratio of the number of scores actually made to the ratio of the total number possible. Table I., below, gives these ratios for every problem for each of the four classes. It shows wide variation in the result of the classes, the largest interval being 32 and the smallest 11,

TABLE I.

Problems.	1	2	3	4	5	6	7	8
Class I	96	85	94	94	98	88	78	81
Class II	83	71	85	79	98	64	64	59
Class III	64	86	67	75	87	66	56	—
Class IV	92	69	77	81	90	71	56	62
All classes	88	71	84	86	96	83	64	69
Variations	32	17	27	19	11	24	22	22

The table should be read as follows. Class I. in problem I. had an attainment of 96, etc.

Notice that the results in the various classes show wide variations, the largest being in problem 1 and the smallest in

problem 5. Class I. makes the best showing and class III. the poorest. An explanation of this difference may be found by securing collaborative evidence from the teachers. With a little more teaching of problems like 2, 7, and 8 class I. seems to be prepared to go on with the course, while class III. might as well study the whole chapter a second time.

If the problems are now ranked in order of attainment, table II. it is seen that the variations in *rank* are small.

TABLE II.

Problems.	1	2	3	4	5	6	7	8
Ranks for class I	2	6	3	3	1	5	8	7
" " " II	3	5	$\boxed{2}$	4	1	6	6	8
" " " III	$\boxed{6}$	$\boxed{2}$	4	3	1	5	7	8
" " " IV	1	6	4	3	2	5	8	7
" " all classes	2	6	4	3	1	5	8	7

Thus, problem 5 was very easy for all classes and problem 2 was hard. The remaining problems very nearly kept their positions. The table shows a uniformity in the results of the teaching in the four classes, because the ranks of the problems are practically the same although the classes are apparently not of equal ability.

Of special interest are the three exceptions marked with squares. When the teacher of class II. was asked why his class did so well with problem 3, he said that former experience had taught him that subtraction of polynomials is a difficult process. Therefore he placed great emphasis on this operation in his teaching. He apparently accomplished good results because in the subtraction of polynomials which enters also in problem 7, as part of the long division process, his class was superior to all others, the ratios for this process being respectively 54 per cent., 74 per cent., 38 per cent., 39 per cent. This also brings out the fact that a process is more difficult when it occurs in a new situation than when it occurs alone, —the ratios for this particular process in problem 3 being considerably higher, 94, 85, 67, and 77.

Class III. did not succeed well with problem 1. The

teacher's opinion was that the word "*collecting*" was new to the pupils, the word "*combining*" being used in class. Sometimes the use of an unfamiliar word in a test confuses the pupils, and the results do not indicate at all to what extent they are really able to work the problem.

Further evidence of their ability to solve this problem may be found by studying to what extent the pupils are successful when the same operation occurs elsewhere. For example, terms are to be collected in the third step in problem 7. The ratios for this step for the various classes are 81 per cent., 53 per cent., 23 per cent., and 60 per cent., showing that class III. is deficient in this process even when no instructions as to "collecting" or "combining" are given. In fact, an examination of the papers of the pupils verifies that they did not understand the process, for such mistakes as $a^2 + a^2 = a^4$ were frequent.

On this basis of the results of this study the teacher during the next class hour retaught the process, and similar errors were carefully corrected in the later part of the course.

That this class can be trained with careful teaching to do as good work as the others, or even better work, is seen from the fact that they surpassed all other classes in problem 2.

By arranging the various processes in the test in order of attainment some interesting comparisons can be made. The order is as follows:

TABLE III.

Processes.		Attainment
1. Multiplication of a binomial by a monomial		93
2. Graphical addition of positive and negative numbers		92
3. Combining similar terms		84
4. Subtraction of polynomials		84
5. Multiplication of monomials		83
sign	80	
numerical product	86	
literal product	80	
6. Reduction of fractions		78
sign	72	
numerical part	80	
literal part	78	
7. Multiplication of two polynomials		73
product of monomials	80	
adding polynomials	50	

8. Evaluation			69
	substitution	72	
	product of positive and negative numbers	69	
	combining terms	67	
9. Multiplying positive and negative numbers graphically			65
10. Dividing by a fraction			61
	inversion	67	
	multiplication	62	
	reduction	55	
11. Graphical subtraction			61
12. Division of polynomials			56
	division of monomials	64	
	product of polynomial by monomial	54	
	subtraction of polynomials	50	

The table shows that best results were attained with multiplying binomials by monomials, and the poorest in adding and subtracting polynomials as shown in 7 and 12. Hence, the latter need to be emphasized and used in many situations before they are really understood.

Graphical addition and subtraction were given as experimental problems. These processes are ordinarily used for illustrative purposes and not as an aim in themselves. It is not intended to drill pupils to be able to reproduce them. Yet one of them was retained as well as anything given in the test; the other proved to be difficult.

The results show uniformly that a process, when mastered alone, is not sufficiently understood until it has been mastered in different situations. For example, the product of monomials shows an attainment of 83; in process 7 it shows 80, in evaluation 69, in division 62. Similarly, for reduction of fractions we have an attainment of 78, in process 10 it shows only 55. For combining terms we have 84, but in evaluation only 67. For multiplication of polynomials by monomials we find an attainment of 80 in process 7, and 54 in process 12.

Subtraction of polynomials: alone 84 per cent., in long division 50 per cent. Multiplication of polynomials by a monomial: in multiplication of polynomials 80 per cent.; in long division 54 per cent.

It is evident that when a new process is taught, the previous processes involved offer new difficulties, and deserve considerable special attention. A similar situation is met when

arithmetical computations occur in an algebraic problem. Errors such as $2x \times 3x = 5x^2$ are common. Certainly the pupil knows that $2 \times 3 = 6$ but he must now concentrate on two processes at the same time, and he must learn to do both correctly. A review or drill in arithmetic alone will not help him to avoid such mistakes.

The process of long division standing at the bottom of the list, is a type of problem frequently entirely omitted from a first-year course. Since it serves as an excellent check on the extent to which the more fundamental processes are actually retained and carried over, it seems unwise to omit the teaching of long division of polynomials.

The test here recorded in full shows that an analytical exposition reveals the difficult parts involved in an operation or in a problem. This information may then be used to teach the problem again with an improved technique or in different situations. Moreover, similar records kept of all tests would be valuable to the teacher when he gives the course a second time and ultimately such systematic testing must lead to a well-organized course.

NEW PUBLICATIONS.

The Copernicus of Antiquity (Aristarchus of Samos). By Sir Thomas L. Heath. London, 1920. Pp. iv + 60.

If any teacher of mathematics is looking for a brief survey of the astronomy of the Greeks, with biographies of the greatest workers in this field—most of them mathematicians as well—and if he wishes this survey set forth in a style that will appeal to himself and his pupils, mingling anecdote with historical facts, let him send forthwith to the New York agents (The Macmillan Company) and order this latest product of the prolific pen of Sir Thomas Heath. Here he will find the story of the makers of the one great application of mathematics among the Greeks; he will find this story told as he would like to tell it to his own classes; he will read the book at a single sitting; he will learn much of ancient science; and incidentally he will see what the study of Greek does for a writer of English.

The volume is one of those little handbooks that are issued by the Society for Promoting Christian Knowledge—handbooks that, unfortunately, we have no fund for publishing in this country—and that bring the contributions of some of the best writers of Great Britain to the doors of the humblest cottage. The United States seems to meet the need for inexpensive literature by the publication of low-priced journals, most of them of little value except to the advertiser; England meets it by several series like the one issued by this well-known society, and of which the volume under review is a type. It is a matter of deep regret, if not of chagrin, that we have no endowment for the publication of worthy scientific books that might help to raise the intellectual standards of our people.

Sir Thomas Heath divides the work into two parts, the first dealing with upwards of a dozen astronomers who lived before the time of Aristarchus (c. 280 B.C.), and the second dealing with Aristarchus himself. In language characterized by simplicity, clearness and good taste, the author mingles the human story with a record of the scientific progress of the greatest

nation, intellectually considered, of ancient times—and perhaps of all time. He shows how the Greek understanding of the heavens developed, beginning with the time when Homer spoke of the earth as a flat, circular disk bounded by the river Oceanus that encircled it and flowed back into itself; and when he sang of the Morning Star, the Evening Star, the Pleiades, the Great Bear and other prominent stars and constellations. He tells again the story of Thales falling into the well while watching the stars, being thereupon chided by "a clever and pretty maidservant from Thrace for being so 'eager to know what goes on in the heavens that he could not see what was in front of him, nay, even at his very feet.'" Here, too, the reader will learn how Anaximander took a step in advance of those who had preceded him, asserting that the earth is the center of the universe and is held in position by being equidistant from all the rest of the heavenly bodies, and that the sun and the moon are carried about the earth on hoops. It was he who attempted to find the size of the circles of these two heavenly bodies—the first noteworthy step in the measurement of the solar system. He will also learn of that great figure in Greek mathematics, Pythagoras, whose "most epoch-making discovery was that of the dependence of musical tones on numerical proportions, the octave representing the proportion of 2:3 in length of string at the same tension, the fifth 3:2 and the fourth 4:3." It is he, so tradition tells us, who first maintained that the earth is a sphere, and who first observed that the planets have an independent motion of their own as distinguished from the motion of the fixed stars. The reader will also find the poet merging into the mathematician and the astronomer, not merely in the case of Parmenides, with his systems of wreaths about the sphere of the universe, forming pathways for the sun and moon and stars, but in other cases as well; and will find that history confirms what is so generally felt by teachers of mathematics—that geometry and poetry are always closely allied. He will learn how a mathematician, Anaxagoras (c. 450 B.C.), first propounded the idea that the moon shines by reflected light; that Plato ranked high as an astronomer as well as a geometer and a philosopher; and that the causes of eclipses were correctly stated by various mathematicians who preceded Aristarchus.

In Aristarchus himself the reader will find the Copernicus of the Greeks—that is, the man who first set forth the heliocentric theory of the solar system. Indeed, Copernicus himself acknowledged his debt to Aristarchus with respect to this great discovery. To the teachers of mathematics, however, it is Aristarchus's treatise *On the Sizes and Distances of the Sun and Moon* that will be of chief interest. In this portion of the work Sir Thomas Heath has gone into the mathematics of Aristarchus and has furnished the teacher with a considerable amount of material that can be used to advantage in the classroom.

The work closes with a brief but helpful bibliography.

It is not so much the facts which are set forth as it is the way in which these facts are presented that makes this little book one that the teacher will be glad to own. The facts can be obtained elsewhere, but not the style, and it is the style and the mission of the book that will lead the reader to overlook the quality of paper that the war has rendered inevitable, and to recognize in the Society for Promoting Christian Knowledge a medium for disseminating healthy information that we may well envy.

David Eugene Smith.

Book Review Series—Elementary Algebra. By Murray J. Leventhal. New York City: Globe Book Company. Pp. 56.

Schools which prepare pupils for Regents or College Entrance Examinations have found systematic review courses a necessary phase of their teaching organization. This *Elementary Algebra* by Leventhal brings together in convenient compass well-selected lists of problems, many of which are taken from previous examinations. It seems to meet admirably the purpose for which it was prepared.

Principles and Methods of Teaching Arithmetic. By James Robert Overman. Chicago: Lyons and Carnahan, 1921. Pp. 340.

The character of texts on the teaching of arithmetic has changed noticeably in recent years. More emphasis upon the

measurement of the outcomes of teaching arithmetic (fifty pages in this text), and a much more detailed analysis of the psychology of habituation and problem solving, have replaced the enumeration of mere teaching *devices* and the perorations about the discipline, logic and definitions of the subject. Professor Overman has interpreted the teaching of arithmetic in terms of the accepted psychology and philosophy of elementary education. The fundamental principles underlying the acquisition of skill in computing and the development of ability to solve problems are illustrated by suggestive Lesson Plans which were worked out by the author and his colleagues in the State Normal College at Bowling Green, Ohio.